Giant vortex state in fast rotating Bose-Einstein Condensates

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Model for a rotating Bose-Einstein Condensate: the Gross-Pitaevskii model

The state of the system is obtained minimizing the Gross-Pitaevskii energy functional:

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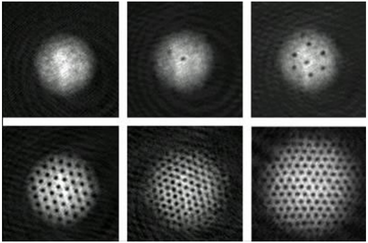

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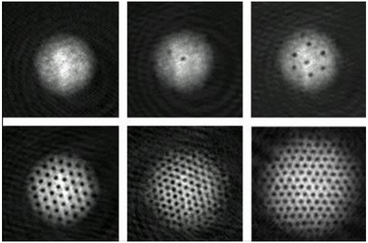
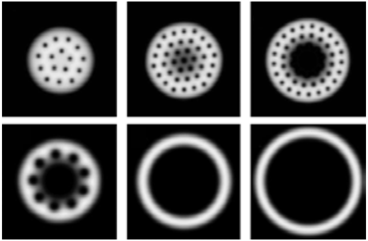
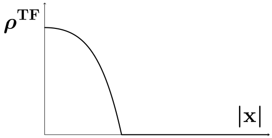




First regime: $\Omega_{\mathrm{rot}} \ll \varepsilon^{-\frac{s-2}{s+2}}$

- in this regime the *mass distribution* of the condensate is well represented by the minimizer of a Thomas-Fermi functional and thus it is essentially supported around the origin;



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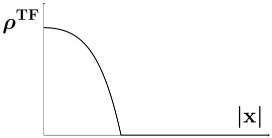

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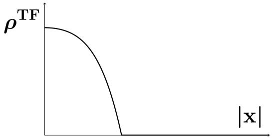
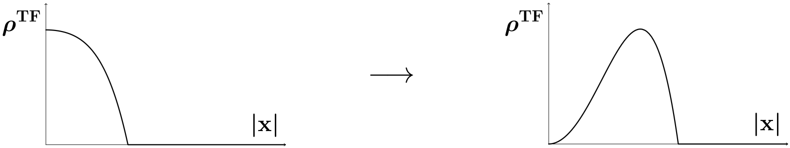


- in particular while $\Omega_{\mathrm{rot}} \ll \varepsilon^{\frac{4}{s+2}}|\log \varepsilon|$ the minimizer has no vortices at all;
- when Ω_{rot} pass a critical value $\Omega_1 \sim \varepsilon^{\frac{4}{s+2}}|\log \varepsilon|$ a first vortex in the origin is nucleated and when $\Omega_{\mathrm{rot}} \gg \varepsilon^{\frac{4}{s+2}}|\log \varepsilon|$ the vortices are uniformly distributed in the whole plane $\mathbb{R}^2$ [first phase transition/first critical speed]

Second regime: $\boxed{\Omega_{\mathrm{rot}} \sim \varepsilon^{-\frac{s-2}{s+2}}}$



In this regime the Thomas-Fermi minimizer digs a hole in the origin, and this has the effect that the mass distribution of the condensate is exponentially small around the origin and becomes essentially supported in an annulus far from the origin; this corresponds to a second critical speed for the condensate of order $\Omega_2 \sim \varepsilon^{-\frac{s-2}{s+2}}$

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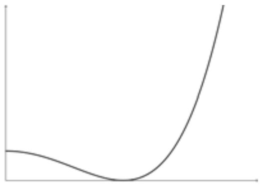
Scaling: the condensate becomes essentially supported in an annulus which radius goes to infinity; we rescale dimension such that the annulus remains at a fixed distance from the origin - that we choose to be 1

$$\mathcal{E}_{\Omega}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left\{ \frac{1}{2} \left| (\nabla - i\mathbf{A}_{\Omega})\, \psi \right|^2 + \Omega^2 W(\mathrm{x}) |\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\}$$

$$W(\mathrm{x}) = \frac{\mathrm{x}^s - 1}{s} - \frac{\mathrm{x}^2 - 1}{2}$$



Second regime: $\Omega \sim \varepsilon^{-1}$

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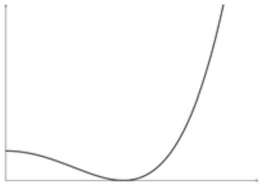
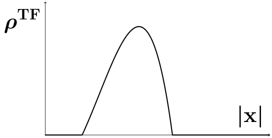


Third regime: $\boxed{\Omega \gg \varepsilon^{-1}}$

- while $\varepsilon^{-1} \ll \Omega \ll \varepsilon^{-4}$ the mass distribution of the condensate is still close to the minimizer of another Thomas-Fermi functional, supported in an annulus of width $\sim (\varepsilon\Omega)^{-\frac{2}{3}}$ going to 0 as $\varepsilon \to 0$;



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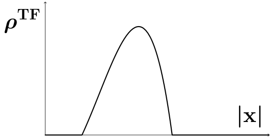

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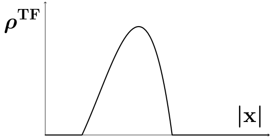

- moreover as long as $\Omega \ll \varepsilon^{-4}$ the minimizer has still vortices distributed in the whole $\mathbb{R}^2$;
- as soon as Ω pass a critical value $\Omega_3 \sim \varepsilon^{-4}$ the vortices in the annulus disappear: third phase transition, minimizer is in the so-called **Giant Vortex state**

For the first two critical speed the exact values are known:

$$\Omega_1 = \frac{\pi}{2}\left(\frac{2k(s+2)}{\pi s}\right)^{\frac{s}{s+2}} \varepsilon^{\frac{4}{s+2}} |\log \varepsilon|,$$

$$\Omega_2 = (2k)^{\frac{s}{s+2}} \left[\frac{4(s+2)}{\pi(s-2)}\right]^{\frac{s-2}{2(s+2)}} \varepsilon^{-\frac{s-2}{s+2}},$$

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Theorem (M. Correggi, N.Rougerie, F. Pinsker, J. Yngvason, 2012)

It exists a constant Ω_c such that if Ω is of the form $\frac{\Omega_0}{\varepsilon^4}$ in the rescaled variables with Ω_0 fixed as $\varepsilon \to 0$ and $\Omega_0 > \Omega_c$ then no minimizer has a zero in the annulus. More precisely, exists a real radial strictly positive function f such that in the annulus we have that

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$$\Omega_{\mathrm{c}} = ?$$

Giant Vortex phase

When $\Omega \gtrsim \varepsilon^{-4}$ any minimizer of the Gross-Pitaevskii energy functional behaves like a real radial function that doesn't vanish in the annulus times a phase factor of degree α around the origin, namely

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$$E_{\Omega}^{\mathrm{GP}} \leq E_{\alpha}^{\mathrm{gv}} \ \forall\alpha \Longrightarrow E_{\Omega}^{\mathrm{GP}} \leq \inf_{\alpha} E_{\alpha}^{\mathrm{gv}} = E_{\alpha_*}^{\mathrm{gv}}$$

$$\alpha_* = \Omega\left(1 + \mathcal{O}(\varepsilon^4)\right)$$

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Main result (M. Correggi, D. D., 2015)

If $\Omega_0 \geq \Omega_c$, with Ω_c explicit, then

$$E^{\mathrm{GP}} = E^{\mathrm{gv}}_{\alpha_*} \left(1 + \mathcal{O}\left(\varepsilon^4\right)\right)$$

and moreover no minimizer of the Gross-Pitaevskii energy functional has a zero in the annulus

- while $\Omega_0 \geq \Omega_c$ there are no vortices in the annulus so the phase transition actually happened and this means that $\Omega_3 \leq \frac{\Omega_c}{\varepsilon^4}$;

- while $\Omega_0 \geq \Omega_\mathrm{c}$ there are no vortices in the annulus so the phase transition actually happened and this means that $\Omega_3 \leq \frac{\Omega_\mathrm{c}}{\varepsilon^4}$;
- it is still an open problem to show that the value we just found is in fact the real critical speed for the condensate, i. e. that in fact $\Omega_3 = \frac{\Omega_\mathrm{c}}{\varepsilon^4}$, and to prove this means to show that for any $\Omega < \frac{\Omega_\mathrm{c}}{\varepsilon^4}$ there are vortices inside the annulus; the minimality argument for α_* gives us hope that the value we found is the critical one

Thanks for the attention!