Critical Speeds for Rotating Bose-Einstein Condensates

Mathematical Foundations of Physics
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this presentation available on daniele.dimonte.it
based on a joint work with Michele Correggi
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In the case of an $N$-body particle system, when $N \to +\infty$, if we assume condensation the state of the system can be described minimizing the following Gross-Pitaevskii energy functional:

$$E_{\omega}^{\text{phys}}[\psi] = \inf_{\|\psi\|_2^2=1} \left\{ \int_{\mathbb{R}^2} \mathrm{d}r \left\{ \frac{1}{2} |\nabla \psi|^2 - \omega \psi^* L_z \psi + \frac{1}{\varepsilon^2} V(r)|\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\} \right\}$$

$$E_{\omega}^{\text{phys}} = \inf_{\|\psi\|_2^2=1} E_{\omega}^{\text{phys}}[\psi] = E_{\omega}^{\text{phys}}[\psi_{\omega}^{\text{phys}}]$$

- $V(r) = r^s$, $s > 2$ is the trapping potential
- $L_z = -i (r \times \nabla)_z = -i \partial_\theta$ angular momentum
- $\omega$ is the rotational speed of the condensate
- $\varepsilon^{-2}$ in front of the quartic term is proportional to the 2-body scattering length for the interacting potential
- asymptotic $\varepsilon \to 0$ (corresponding to the Thomas-Fermi regime)
- asymptotic for $\omega$ as $\varepsilon$ goes to 0
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Increasing the rotational speed $\omega$ we can observe different behaviors:

- formation of quantized vortices in the condensate (related to superfluidity properties): change of the phase of $\Psi^\text{phys}_\omega$
- different shapes of the condensate: change of the modulus of $\Psi^\text{phys}_\omega$

$\psi$ has a vortex in $x_0$ if around the point

$$\psi(x) \sim e^{in\theta(x)} f(|x - x_0|)$$

Numerics by K. Kasamatsu, M. Tsubota, M. Ueda
First regime: $\omega \ll \varepsilon^{-1}$

- When $\omega = 0$ the minimizer is radial and its support is concentrated in an area around a ball centered in zero, and we can only see the effect of the trapping potential.
- For small rotations $\omega < \omega_{c_1} = \omega_1 |\log \varepsilon|$ we can observe no effect on the state of the system; in particular $E_{\omega}^{\text{phys}} = E_0^{\text{phys}}$ and $\psi_{\omega}^{\text{phys}} = \psi_0^{\text{phys}}$ (Aftalion, Jerrard, Royo-Letelier, [AJR11])
- As soon as $\omega \geq \omega_{c_1}$ vortices start to appear in the bulk of the condensate (Ignat, Millot, [IM06]) and when $\omega \gg |\log \varepsilon|$ the vortices are distributed uniformly (Correggi, Yngvason, [CY08])

Experiments from the Cornell Group, Jila research center
Increasing the Rotational Speed

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Experiments from the Cornell Group, Jila research center
Second regime: \( \omega \sim \varepsilon^{-1} \)

- While \( \omega \lesssim \omega_c^2 = \omega_0 \varepsilon^{-1} \) the condensate is still contained in a ball around zero and has vortices uniformly distributed in its bulk.
- When \( \omega = \omega_0 \varepsilon^{-1} \) the centrifugal force comes into play and the \( \Psi^\text{phys}_\omega \) becomes exponentially small in \( \varepsilon \) in a region close to the origin (Correggi, Pinsker, Rougerie, Yngvason, [CPRY12]).

Numerics from Fetter, Jackson, Stringari [FJS05]
Increasing the Rotational Speed

Fast Rotation

$$\mathcal{E}_\omega^{\text{phys}}[\Psi] = \int_{\mathbb{R}^2} \text{d}r \left\{ \frac{1}{2} |(\nabla - iA_{\text{rot}}) \Psi|^2 + \frac{1}{\varepsilon^2} \left[ r^s - \frac{1}{2} \varepsilon^2 \omega^2 r^2 + |\Psi|^2 \right] |\Psi|^2 \right\}$$

$$A_{\text{rot}} = \omega r^\perp = \omega (-r_2, r_1)$$

When $\omega \gg \varepsilon^{-1}$ the condensate gets concentrated on a thin annulus of mean radiusequal to the minimum point of $r^s - \frac{1}{2} \varepsilon^2 \omega^2 r^2$ (notice that $s > 2$ is crucial here)

Rescaling the lengths by this radius we obtain

$$\mathcal{E}_\Omega^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \text{d}x \left\{ \frac{1}{2} |(\nabla - iA_\Omega) \psi|^2 + \Omega^2 W(x)|\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\}$$

$$A_\Omega = \Omega x^\perp, \quad W(x) = \frac{x^s - 1}{s} - \frac{x^2 - 1}{2},$$

$$E_\Omega^{\text{GP}} = \inf_{\|\psi\|_2^2 = 1} \mathcal{E}_\Omega^{\text{GP}}[\psi] = \mathcal{E}_\Omega^{\text{GP}}[\psi^{\text{GP}}]$$
Increasing the Rotational Speed

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- The potential \( W \) is positive and has one only minimum in \( x = 1, W(1) = 0 \)
- \( \Omega \) is the rescaled rotational speed
- When \( \omega \sim \varepsilon^{-1} \) also \( \Omega \sim \varepsilon^{-1} \)
Third regime: $\Omega \gg \varepsilon^{-1}$

- When $\Omega_{c_2} \leq \Omega \ll \varepsilon^{-4}$ it was proven in [CPRY12] that $E_{\Omega}^{GP} = E_{\Omega}^{TF} + \mathcal{O} (\Omega \log (\varepsilon^4 \Omega))$, where $E_{\Omega}^{TF}$ is the ground state of

$$
\mathcal{E}_{\Omega}^{TF} [\rho] = \int_{\mathbb{R}^2} d\mathbf{x} \left[ \frac{1}{\varepsilon^2} \rho + \Omega^2 W(x) \right] \rho
$$

- Using the energy asymptotic it was possible to show also that the profile of the minimizer is exponentially small in $\varepsilon$ outside a ring of radius 1 and of width $(\varepsilon \Omega)^{-\frac{2}{3}} = o(1)$

- Moreover, using the same asymptotic it is also possible to prove that the distribution of vorticity is uniform for $\Omega_{c_2} \leq \Omega \ll \varepsilon^{-4}$
Increasing the Rotational Speed

- When $\Omega = \Omega_0 \varepsilon^{-4}$ the size of a single vortex becomes comparable to the width of the annulus where $\psi_{\Omega}^{\text{GP}}$ is essentially supported.
- Another transition occurs and vortices are expelled from the bulk of the condensate (Giant Vortex state).

**Theorem** (Correggi, Pinsker, Rougerie, Yngvason, [CPRY12])

If $\Omega_0$ is big enough then $\psi_{\Omega}^{\text{GP}}$ has no zeroes in the annulus, and therefore there are no vortices in the annulus; more precisely

$$|\psi_{\Omega}^{\text{GP}}(x)| = \frac{1}{\sqrt{2\pi x}} g_{\text{gv}}(x) (1 + o(1)), \quad g_{\text{gv}} > 0$$
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Numerics from Fetter, Jackson, Stringari,[FJS]
An essential tool in [CPRY12] was proving the approximation of the energy in terms of a Giant Vortex energy; assuming $\Omega_0 \gg 1$

$$E_{\Omega}^{GP} = \frac{E^{gv}}{\varepsilon^4} + \mathcal{O} \left( |\log \varepsilon|^{\frac{9}{2}} \right)$$

$$\mathcal{E}^{gv}[g] = \int_{\mathbb{R}} dy \left\{ \frac{1}{2} |\nabla g| + \frac{1}{2} \Omega_0^2 (s + 2)y^2 g^2 + \frac{1}{2\pi} g^4 \right\}$$

$$E^{gv} = \inf_{\|g\|_2^2} \mathcal{E}^{gv}[g] = \mathcal{E}^{gv}[g_{gv}]$$
In particular the minimizer $g_{gv}$ is strictly positive in the annulus $A$ where $\psi^{GP}_\Omega$ is concentrated, so we can define $u$ such that $\psi^{GP}_\Omega(x) = \frac{1}{\sqrt{2\pi\varepsilon}} g_{gv}(x) u(x) e^{i[\Omega] \theta}$, and in this case

$$\frac{E^{gv}}{\varepsilon^4} \geq E^{GP}_\Omega \geq \frac{E^{gv}}{\varepsilon^4} + \frac{1}{2\pi\varepsilon^2} \int_A \text{d}x \ K(x) |\nabla u|^2 + O \left( |\log \varepsilon|^{\frac{9}{2}} \right)$$

and the lower bound of this estimate reduces to estimate the positivity of the cost function; using the fact that for $\Omega_0$ large enough the profile of $g_{gv}$ is basically gaussian, they were able to prove

$$K(x) \geq C \left( 1 + O \left( \Omega_0^{-\frac{1}{4}} \right) \right) g^{2}_{gv}(x)$$

In [CPRY12] no explicit value for the transition speed was given, due to the crucial hypotheses $\Omega_0 \gg 1$.
In particular the minimizer $g_{gv}$ is strictly positive in the annulus $A$ where $\psi_{\Omega}^{GP}$ is concentrated, so we can define $u$ such that

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Main Result

Critical velocity (Correggi, D, [CD15])

Let $\Omega_c$ be defined as the supremum of the solution of

$$
\Omega_0 = \frac{4}{s + 2} \left[ \mu_{gv} - \frac{1}{2\pi} g_{gv}^2(0) \right]
$$

where $\mu_{gv}$ is defined through

$$
-\frac{1}{2} g_{gv}'' + \frac{1}{2} \Omega_0^2 (s + 2) y^2 g_{gv} + \frac{1}{\pi} g_{gv}^3 = \mu_{gv} g_{gv} ;
$$

then if $\Omega_0 \geq \Omega_c$ then

$$
E^{GP}_\Omega = \frac{E^{gv}}{\varepsilon^4} + O(1)
$$

This value for the critical speed is extracted proving the positivity of $K$ and in particular that $K(x) > 0$ in $A$ if and only if $K(0) > 0$.
It is possible to prove that there is a solution to the equation that defines $\Omega_c$ by considering the limits for $\Omega_0$ going both to 0 and to $+\infty$; we expect this solution to be unique.

As in [CPRY12] the main consequence of this estimate is that $\psi_{\Omega}^{GP}$ is strictly positive in $A$ and therefore the condensate has no vortices in $A$.

Therefore while $\Omega_0 \geq \Omega_c$ there are no vortices in the annulus so the phase transition already happened and this means that $\Omega_{c3} \leq \frac{\Omega_c}{\epsilon^4}$.

It is still an open problem to show that the value we just found is in fact the real critical speed for the condensate, that is that in fact $\Omega_{c3} = \frac{\Omega_c}{\epsilon^4}$, and to prove this one should show that for any $\Omega < \frac{\Omega_c}{\epsilon^4}$ there are vortices inside the annulus.
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Thanks for the attention!

Non-Existence of Vortices in the Small Density Region of a Condensate (2011)

[IM06] R. Ignat, V. Millot
Energy Expansion and Vortex Location for a Two Dimensional Rotating Bose-Einstein Condensate (2006)

[CY08] M. Correggi, J. Yngvason
Energy and Vorticity in Fast Rotating Bose-Einstein Condensates (2008)

Critical Rotational Speeds for Superfluids in Homogeneous Traps (2012)

[FJS05] A.L. Fetter, N. Jackson, S. Stringari
Rapid Rotation of a Bose-Einstein Condensate in a Harmonic Plus Quartic Trap (2005)

[CD16] M. Correggi, D. Dimonte
On the third Critical Speed for Rotating Bose-Einstein Condensates (2016)