Critical Speeds for Rotating Bose-Einstein Condensates

Mathematical Foundations of Physics

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this presentation available on `daniele.dimonte.it`
based on a joint work with **Michele Correggi**

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In the case of an N-body particle system, when $N \to +\infty$, if we assume condensation the state of the system can be described minimizing the following Gross-Pitaevskii energy functional:

$$\mathcal{E}_\omega^{\mathrm{phys}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \left\{ \frac{1}{2} |\nabla \Psi|^2 - \omega \Psi^* L_z \Psi + \frac{1}{\varepsilon^2} V(\mathbf{r}) |\Psi|^2 + \frac{1}{\varepsilon^2} |\Psi|^4 \right\}$$

$$E_\omega^{\mathrm{phys}} = \inf_{\|\Psi\|_2^2 = 1} \mathcal{E}_\omega^{\mathrm{phys}}[\Psi] = \mathcal{E}_\omega^{\mathrm{phys}}[\Psi_\omega^{\mathrm{phys}}]$$

- $V(\mathbf{r}) = r^s$, $s > 2$ is the trapping potential
- $L_z = -i\left(\mathbf{r} \times \nabla\right)_z = -i\partial_\theta$ angular momentum
- ω is the rotational speed of the condensate
- ε^{-2} in front of the quartic term is proportional to the 2-body scattering length for the interacting potential
- asymptotic $\varepsilon \to 0$ (corresponding to the Thomas-Fermi regime)
- asymptotic for ω as ε goes to 0

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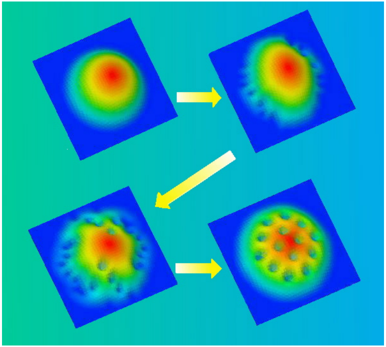
Increasing the rotational speed ω we can observe different behaviors:

- formation of quantized vortices in the condensate (related to superfluidity properties): change of the phase of $\Psi_\omega^{\mathrm{phys}}$
- different shapes of the condensate: change of the modulus of $\Psi_\omega^{\mathrm{phys}}$



Numerics by K. Kasamatsu, M. Tsubota, M. Ueda

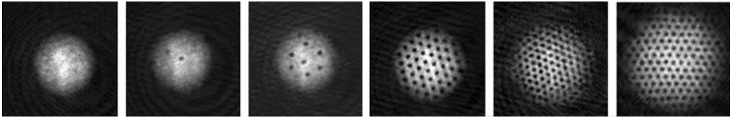
ψ has a vortex in x_0

if around the point

$$\psi(x) \simeq e^{in\theta(x)} f\left(|x - x_0|\right)$$

First regime: $\omega \ll \varepsilon^{-1}$

- When $\omega = 0$ the minimizer is radial and its support is concentrated in an area around a ball centered in zero, and we can only see the effect of the trapping potential.
- For small rotations $\omega < \omega_{c_1} = \omega_1 |\log \varepsilon|$ we can observe no effect on the state of the system; in particular $E_\omega^{\text{phys}} = E_0^{\text{phys}}$ and $\Psi_\omega^{\text{phys}} = \Psi_0^{\text{phys}}$ (Aftalion, Jerrard, Royo-Letelier, [AJR11])
- As soon as $\omega \geq \omega_{c_1}$ vortices start to appear in the bulk of the condensate (Ignat, Millot, [IM06]) and when $\omega \gg |log \varepsilon|$ the vortices are distributed uniformly (Correggi, Yngvason, [CY08])



Experiments from the Cornell Group, Jila research center

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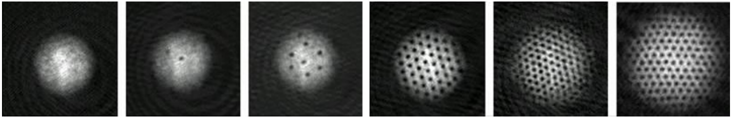

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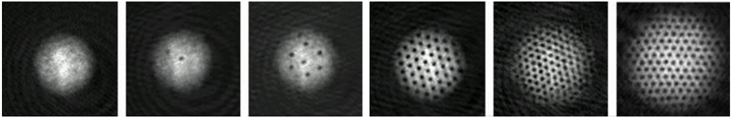

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Second regime: $\boxed{\omega \sim \varepsilon^{-1}}$

- While $\omega \lesssim \omega_{c_2} = \omega_2 \varepsilon^{-1}$ the condensate is still contained in a ball around zero and has vortices uniformly distributed in its bulk
- When $\omega = \omega_0 \varepsilon^{-1}$ the centrifugal force comes into play and the $\Psi_\omega^{\mathrm{phys}}$ becomes exponentially small in ε in a region close to the origin (Correggi, Pinsker, Rougerie, Yngvason, [CPRY12])

Numerics from Fetter, Jackson, Stringari [FJS05]

$$\mathcal{E}_{\omega}^{\mathrm{phys}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \left\{ \frac{1}{2} \left| \left(\nabla - i\mathbf{A}_{\mathrm{rot}} \right) \Psi \right|^2 + \frac{1}{\varepsilon^2} \left[r^s - \frac{1}{2}\varepsilon^2 \omega^2 r^2 + |\Psi|^2 \right] |\Psi|^2 \right\}$$

$$\mathbf{A}_{\mathrm{rot}} = \omega \mathbf{r}^{\perp} = \omega \left(-r_2, r_1 \right)$$

When $\omega \gg \varepsilon^{-1}$ the condensate gets concentrated on a thin annulus of mean radiusequal to the minimum point of $r^s - \frac{1}{2}\varepsilon^2\omega^2 r^2$ (notice that $s > 2$ is crucial here)

Rescaling the lengths by this radius we obtain

$$\boxed{\mathcal{E}_{\Omega}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left\{ \frac{1}{2} \left| \left(\nabla - i\mathbf{A}_{\Omega} \right) \psi \right|^2 + \Omega^2 W(\mathrm{x}) |\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\}}$$

$$\mathbf{A}_{\Omega} = \Omega \mathbf{x}^{\perp}, \quad W(\mathrm{x}) = \frac{\mathrm{x}^s - 1}{s} - \frac{x^2 - 1}{2},$$

$$E_{\Omega}^{\mathrm{GP}} = \inf_{\|\psi\|_2^2 = 1} \mathcal{E}_{\Omega}^{\mathrm{GP}}[\psi] = \mathcal{E}_{\Omega}^{\mathrm{GP}}[\psi_{\Omega}^{\mathrm{GP}}]$$

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- The potential W is positive and has one only minimum in $\mathrm{x} = 1$, $W(1) = 0$
- Ω is the rescaled rotational speed
- When $\omega \sim \varepsilon^{-1}$ also $\Omega \sim \varepsilon^{-1}$

Third regime: $\Omega \gg \varepsilon^{-1}$

- When $\Omega_{c_2} \leq \Omega \ll \varepsilon^{-4}$ it was proven in [CPRY12] that $E_\Omega^{\mathrm{GP}} = E_\Omega^{\mathrm{TF}} + \mathcal{O}\left(\Omega \left|\log(\varepsilon^4 \Omega)\right)\right)$, where E_Ω^{TF} is the ground state of

$$\mathcal{E}_\Omega^{\mathrm{TF}}[\rho] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left[\frac{1}{\varepsilon^2}\rho + \Omega^2 W(\mathrm{x})\right] \rho$$

- Using the energy asymptotic it was possible to show also that the profile of the minimizer is exponentially small in ε outside a ring of radius 1 and of width $(\varepsilon\Omega)^{-\frac{2}{3}} = o(1)$
- Moreover, using the same asymptotic it is also possible to prove that the distribution of vorticity is uniform for $\Omega_{c_2} \leq \Omega \ll \varepsilon^{-4}$

- When $\Omega = \Omega_0 \varepsilon^{-4}$ the size of a single vortex becomes comparable to the width of the annulus where $\psi_\Omega^{\mathrm{GP}}$ is essentially supported
- Another transition occurs and vortices are expelled from the bulk of the condensate (**Giant Vortex state**)

Theorem (Correggi, Pinsker, Rougerie, Yngvason, [CPRY12])

If Ω_0 is big enough then $\psi_\Omega^{\mathrm{GP}}$ has no zeroes in the annulus, and therefore there are no vortices in the annulus; more precisely

$$\left|\psi_\Omega^{\mathrm{GP}}(\mathbf{x})\right| = \frac{1}{\sqrt{2\pi x}} g_{\mathrm{gv}}(x) \left(1 + o\left(1\right)\right), \quad g_{\mathrm{gv}} > 0$$

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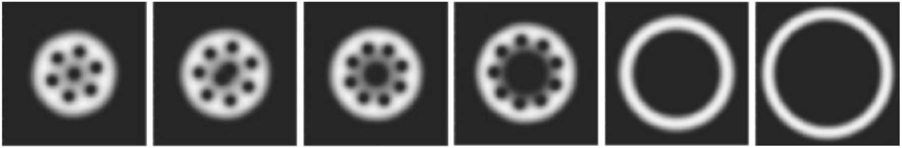

Numerics from Fetter, Jackson, Stringari,[FJS]

An essential tool in [CPRY12] was proving the approximation of the energy in terms of a Giant Vortex energy; assuming $\Omega_0 \gg 1$

$$E_\Omega^{\mathrm{GP}} = \frac{E^{\mathrm{gv}}}{\varepsilon^4} + \mathcal{O}\left(|\log \varepsilon|^{\frac{9}{2}}\right)$$

$$\mathcal{E}^{\mathrm{gv}}[g] = \int_{\mathbb{R}} \mathrm{dy} \left\{ \frac{1}{2} |\nabla g| + \frac{1}{2} \Omega_0^2 (s+2) \mathrm{y}^2 g^2 + \frac{1}{2\pi} g^4 \right\}$$

$$E^{\mathrm{gv}} = \inf_{\|g\|_2^2} \mathcal{E}^{\mathrm{gv}}[g] = \mathcal{E}^{\mathrm{gv}}[g_{\mathrm{gv}}]$$

In particular the minimizer g_{gv} is strictly positive in the annulus A where $\psi_{\Omega}^{\mathrm{GP}}$ is concentrated, so we can define u such that $\psi_{\Omega}^{\mathrm{GP}}(x) = \frac{1}{\sqrt{2\pi\varepsilon}} g_{\mathrm{gv}}(x) u(x) e^{i\lfloor\Omega\rfloor\theta}$, and in this case

$$\frac{E^{\mathrm{gv}}}{\varepsilon^4} \geq E_{\Omega}^{\mathrm{GP}} \geq \frac{E^{\mathrm{gv}}}{\varepsilon^4} + \frac{1}{2\pi\varepsilon^2}\int_A \mathrm{d}\mathbf{x}\ K(\mathrm{x})\,|\nabla u|^2 + \mathcal{O}\left(|\log\varepsilon|^{\frac{9}{2}}\right)$$

and the lower bound of this estimate reduces to estimate the positivity of the cost function; using the fact that for Ω_0 large enough the profile of g_{gv} is basically gaussian, they were able to prove

$$K(x) \geq C\left(1 + \mathcal{O}\left(\Omega_0^{-\frac{1}{4}}\right)\right) g_{\mathrm{gv}}^2(\mathrm{x})$$

In [CPRY12] no explicit value for the transition speed was given, due to the crucial hypotheses $\Omega_0 \gg 1$

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Main Result

Critical velocity (Correggi, D, [CD15])

Let Ω_c be defined as the supremum of the solution of

$$\boxed{\Omega_0 = \frac{4}{s+2}\left[\mu^{\mathrm{gv}} - \frac{1}{2\pi} g_{\mathrm{gv}}^2(0)\right]}$$

where μ^{gv} is defined through $-\frac{1}{2}g_{\mathrm{gv}}'' + \frac{1}{2}\Omega_0^2(s+2)\mathrm{y}^2 g_{\mathrm{gv}} + \frac{1}{\pi}g_{\mathrm{gv}}^3 = \mu^{\mathrm{gv}} g_{\mathrm{gv}}$; then if $\Omega_0 \geq \Omega_c$ then

$$E_\Omega^{\mathrm{GP}} = \frac{E^{\mathrm{gv}}}{\varepsilon^4} + \mathcal{O}(1)$$

This value for the critical speed is extracted proving the positivity of K and in particular that $K(\mathrm{x}) > 0$ in A if and only if $K(0) > 0$

- It is possible to prove that there is a solution to the equation that defines Ω_c by considering the limits for Ω_0 going both to 0 and to $+\infty$; we espect this solution to be unique
- As in [CPRY12] the main consequence of this estimate is that $\psi_\Omega^{\mathrm{GP}}$ is strictly positive in A and therefore the condensate has no vortices in A
- Therefore while $\Omega_0 \geq \Omega_c$ there are no vortices in the annulus so the phase transition already happened and this means that $\Omega_{c_3} \leq \frac{\Omega_c}{\varepsilon^4}$;
- it is still an open problem to show that the value we just found is in fact the real critical speed for the condensate, that is that in fact $\Omega_{c_3} = \frac{\Omega_c}{\varepsilon^4}$, and to prove this one should show that for any $\Omega < \frac{\Omega_c}{\varepsilon^4}$ there are vortices inside the annulus

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Thanks for the attention!

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[CD16] M. Correggi, D. Dimonte
On the third Critical Speed for Rotating Bose-Einstein Condensates (2016)