Energy asymptotics for a fast rotating Bose-Einstein Condensate

AMPQ Seminar
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this presentation is available on daniele.dimonte.it
based on a joint work with Michele Correggi
1. Mathematical Framework
   - Model for a rotating BEC: The Gross-Pitaevskii model

2. Increasing the Rotational Speed
   - Slow Rotation
   - Fast Rotation
   - Ultrarapid Rotation
   - Giant Vortex state

3. The third critical speed
   - Sketch of the proof
   - An explicit value
   - Conclusion and perspectives
In the case of an $N$-body particle system, when $N \to +\infty$, if we assume condensation the state of the system can be described minimizing the following Gross-Pitaevskii energy functional:

$$E_{\omega}^{\text{phys}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}r \left\{ \frac{1}{2} |\nabla \Psi|^2 - \omega \Psi^* L_z \Psi + \frac{1}{\varepsilon^2} V(r)|\Psi|^2 + \frac{1}{\varepsilon^2} |\Psi|^4 \right\}$$

$$E_{\omega}^{\text{phys}} = \inf_{\|\Psi\|^2_2=1} E_{\omega}^{\text{phys}}[\Psi] = E_{\omega}^{\text{phys}}[\Psi_\omega]$$

- $V(r) = r^s$, $s > 2$ is the trapping potential
- $L_z = -i (r \times \nabla)_z = -i \partial_\theta$ angular momentum
- $\omega$ is the rotational speed of the condensate
- $\varepsilon^{-2}$ in front of the quartic term is proportional to the 2-body scattering length for the interacting potential
- asymptotic $\varepsilon \to 0$ (corresponding to the Thomas-Fermi regime)
- asymptotic for $\omega$ as $\varepsilon$ goes to 0
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Increasing the rotational speed $\omega$ we can observe different behaviors:

- formation of quantized vortices in the condensate (related to superfluidity properties): change of the phase of $\Psi^\text{phys}_\omega$
- different shapes of the condensate: change of the modulus of $\Psi^\text{phys}_\omega$

\[ \psi \text{ has a vortex in } x_0 \]

if around the point

\[ \psi(x) \sim e^{in\theta(x)} f(|x - x_0|) \]

Numerics by K. Kasamatsu, M. Tsubota, M. Ueda
First regime: $\omega \ll \varepsilon^{-1}$

- When $\omega = 0$ the minimizer can be taken as a strictly positive function (up to a $U(1)$ symmetry) and it’s close to the minimizer of a Thomas-Fermi functional; its support is concentrated in an area around a ball centered in zero, and we can only see the effect of the trapping potential.

- For small rotations $\omega < \omega_{c1} = \omega_1 |\log \varepsilon|$ we can observe no effect on the state of the system; in particular $E^\text{phys}_\omega = E^\text{phys}_0$ and $\Psi^\text{phys}_\omega = \Psi^\text{phys}_0$ (Aftalion, Jerrard, Royo-Letelier, [AJR11]).

Experiments from the Cornell Group, Jila research center.
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Experiments from the Cornell Group, Jila research center.
Increasing the Rotational Speed

First regime: $\omega \ll \varepsilon^{-1}$

- As soon as $\omega \geq \omega_{c_1}$ vortices start to appear in the bulk of the condensate (Ignat, Millot, [IM06]) showing the contribute from the kinetic term
- As the second vortex appear the rotational symmetry is broken in the minimizer and is never restored, even for higher rotational speeds
- Increasing the rotational speed even more, for $\omega \gg |\log \varepsilon|$ the vortices become distributed uniformly (Correggi, Yngvason, [CY08])

Experiments from the Cornell Group, Jila research center
**Second regime:**  \( \omega \sim \varepsilon^{-1} \)

- While \( \omega \lesssim \omega_2 = \omega_2 \varepsilon^{-1} \) the condensate is still contained in a ball around zero and has vortices uniformly distributed in its bulk.
- When \( \omega = \omega_0 \varepsilon^{-1} \) the Thomas-Fermi functional digs a hole in the origin, therefore the \( \Psi_\omega^{\text{phys}} \) becomes exponentially small in \( \varepsilon \) in a region close to the origin (Correggi, Pinsker, Rougerie, Yngvason, [CPRY12]).

Numerics from Fetter, Jackson, Stringari [FJS05]
Increasing the Rotational Speed

Fast Rotation

\[ \mathcal{E}^{\text{phys}}_\omega [\Psi] = \int_{\mathbb{R}^2} \text{d}r \left\{ \frac{1}{2} |(\nabla - iA_{\text{rot}}) \Psi|^2 + \frac{1}{\varepsilon^2} \left[ r^s - \frac{1}{2} \varepsilon^2 \omega^2 r^2 + |\Psi|^2 \right] |\Psi|^2 \right\} \]

\[ A_{\text{rot}} = \omega r^\perp = \omega (-r_2, r_1) \]

When \( \omega \gg \varepsilon^{-1} \) the support of the Thomas-Fermi profile becomes concentrated in an annulus of mean radius equal to the minimum point of \( r^s - \frac{1}{2} \varepsilon^2 \omega^2 r^2 \), which is going to infinity as \( \varepsilon \) goes to zero.

Here the hypotheses of \( s > 2 \) becomes crucial, and if \( s = 2 \) as soon as \( \omega \gg \varepsilon^{-1} \) the energy becomes unbounded from below.
Increasing the Rotational Speed

Fast Rotation

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When \( \omega \gg \varepsilon^{-1} \) the support of the Thomas-Fermi profile becomes concentrated in an annulus of mean radius equal to the minimum point of \( r^s - \frac{1}{2} \varepsilon^2 \omega^2 r^2 \)

We rescale the radius around this minimum point in such a way that

\[ E_{\Omega}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \text{d}x \left\{ \frac{1}{2} \left| \nabla - iA_{\Omega} \right| \psi \right|^2 + \Omega^2 W(x) |\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\} \]

\[ A_{\Omega} = \Omega x^\perp, \quad W(x) = \frac{x^s - 1}{s} - \frac{x^2 - 1}{2}, \]

\[ E_{\Omega}^{\text{GP}} = \inf_{\|\psi\|_2^2 = 1} E_{\Omega}^{\text{GP}}[\psi] = E_{\Omega}^{\text{GP}}[\psi_{\Omega}^{\text{GP}}] \]
Increasing the Rotational Speed

Fast Rotation

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- The potential \( W \) is positive and has one only minimum in \( x = 1, W(1) = 0 \)
- \( \Omega \) is the rescaled rotational speed
- When \( \omega \sim \varepsilon^{-1} \) also \( \Omega \sim \varepsilon^{-1} \)
Increasing the Rotational Speed

Third regime: $\Omega \gg \varepsilon^{-1}$

- When $\Omega_{c2} \leq \Omega \ll \varepsilon^{-4}$ it was proven in [CPRY12] that $E_{\Omega}^{GP} = E_{\Omega}^{TF} + \mathcal{O} (\Omega \left| \log (\varepsilon^4 \Omega) \right| )$, where $E_{\Omega}^{TF}$ is the ground state of

$$
E_{\Omega}^{TF}[\rho] = \int_{\mathbb{R}^2} d\mathbf{x} \left[ \Omega^2 W(x) + \frac{1}{\varepsilon^2 \rho} \right] \rho
$$

- Using the energy asymptotic it was possible to show also that the profile of the minimizer is exponentially small in $\varepsilon$ outside a ring of radius 1 and of width $(\varepsilon \Omega)^{-\frac{2}{3}} = o(1)$

- Moreover, using the same asymptotic it is also possible to prove that the distribution of vorticity is uniform for $\Omega_{c2} \leq \Omega \ll \varepsilon^{-4}$
When $\Omega = \Omega_0 \varepsilon^{-4}$ the size of a single vortex becomes comparable to the width of the annulus where $\psi_{\Omega}^{\text{GP}}$ is essentially supported.

Indeed below the threshold the size of a single vortex, that is the size of the region where $|\psi_{\Omega}^{\text{GP}}|^2 \sim 0$, has a core radius $\rho$ which is fixed by the energy to be of order $\rho \sim \Omega_0^{-\frac{1}{3}} \varepsilon^2$.

At the same time the width of the annulus is $(\varepsilon \Omega)^{-\frac{2}{3}} \sim \Omega_0^{-\frac{2}{3}} \varepsilon^2$.

For $\Omega_0$ big enough the vortices don’t fit in the annulus anymore and are expelled from the bulk of the condensate (Giant Vortex state).
Theorem (Correggi, Pinsker, Rougerie, Yngvason, [CPRY12])

If $\Omega_0$ is big enough then $\psi_{\Omega}^{GP}$ has no zeroes in the annulus, and therefore there are no vortices in the annulus; more precisely

$$|\psi_{\Omega}^{GP}(x)| = \frac{1}{\sqrt{2\pi \varepsilon}} g_{gv}(x) (1 + o(1)), \quad g_{gv} > 0$$

Numerics from Fetter, Jackson, Stringari, [FJS]
Increasing the Rotational Speed

An essential tool in [CPRY12] was proving the approximation of the energy in terms of a Giant Vortex energy; assuming $\Omega_0 \gg 1$

$$E_{\Omega}^{GP} = \frac{E^{gv}}{\varepsilon^4} + \mathcal{O} \left( |\log \varepsilon|^{\frac{9}{2}} \right)$$

$$\mathcal{E}^{gv}[g] = \int_{\mathbb{R}} dy \left\{ \frac{1}{2} |\nabla g| + \frac{1}{2} \Omega_0^2 (s + 2) y^2 g^2 + \frac{1}{2\pi} g^4 \right\}$$

$$E^{gv} = \inf_{\|g\|_2} \mathcal{E}^{gv}[g] = \mathcal{E}^{gv}[g_{gv}]$$

Notice that this also shows that the shape of the minimizer is now close to a profile different from the Thomas-Fermi one of before.
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$$E^\text{GP}_\Omega = \frac{E^\text{gv}}{\varepsilon^4} + O\left(|\log \varepsilon|^{\frac{9}{2}}\right)$$

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$$E^{\text{gv}} = \inf_{\|g\|_2^2} \mathcal{E}^{\text{gv}}[g] = \mathcal{E}^{\text{gv}}[g_{\text{gv}}]$$

Notice that this also shows that the shape of the minimizer is now close to a profile different from the Thomas-Fermi one of before.
In particular the minimizer $g_{gv}$ is strictly positive in the annulus $A_{bulk}$ where $\psi^{GP}_\Omega$ is concentrated, so we can define $u$ such that

$$
\psi^{GP}_\Omega(x) = \frac{1}{\sqrt{2\pi \epsilon}} g_{gv}(x) u(x) e^{i[\Omega] \theta},
$$

and in this case

$$
\frac{E_{gv}}{\epsilon^4} \geq E^{GP}_\Omega \geq \frac{E_{gv}}{\epsilon^4} + \frac{1}{2\pi \epsilon^2} \int_{A_{bulk}} d\mathbf{x} K(x) |\nabla u|^2 + O \left( |\log \epsilon|^{9/2} \right)
$$

and the lower bound of this estimate reduces to estimate the positivity of the cost function; using the fact that for $\Omega_0$ large enough the profile of $g_{gv}$ is basically gaussian, they were able to prove

$$
K(x) \geq C \left( 1 + O \left( \Omega_0^{-\frac{1}{4}} \right) \right) g_{gv}^2(x)
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In [CPRY12] no explicit value for the transition speed was given, due to the crucial hypotheses $\Omega_0 \gg 1$
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In our work we decided to focus more on the exact form of $K(x)$ to get a better description of the condition to have a lower bound.

Using again a splitting of the energy one can get

$$E_{\Omega}^{GP} = \frac{E_{gv}}{\varepsilon^4} + \frac{1}{2\pi\varepsilon^2} \mathcal{E}[u] + O(\varepsilon^\infty)$$

$$\mathcal{E}[u] := \int_{A_{bulk}} dx \ g_{gv}^2 \left\{ \frac{1}{2} |\nabla u|^2 + a \cdot j[u] + \frac{1}{2\pi\varepsilon^4} g_{gv}^2 \left( 1 - |u|^2 \right)^2 \right\}$$

$$\mathcal{E}[u] \geq \int_{A_{bulk}} dx \ K(x) |\nabla u|^2 + \frac{1}{2\pi\varepsilon^4} g_{gv}^4 \left( 1 - |u|^2 \right)^2$$

$$K(x) := \frac{1}{2} g_{gv}^2 - F(x)$$

Imposing the condition $K(x) > 0$ to get the lower bound gives us a critical value for $\Omega_0$; in particular $K(x) > 0$ in $A_{bulk}$ if and only if $K(0) > 0$. 

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$$\mathcal{E}[u] := \int_{A_{bulk}} dx \ g_{gv}^2 \left\{ \frac{1}{2} |\nabla u|^2 + [a \cdot j[u]] + \frac{1}{2\pi\varepsilon^4} g_{gv}^2 \left( 1 - |u|^2 \right)^2 \right\}$$

$$a(x) := \left( \frac{\Omega + \beta_x}{x} - \Omega x \right) e_\theta, \quad j[u] := \text{Im} (u^* \nabla u)$$

$$\mathcal{E}[u] \geq \int_{A_{bulk}} dx \ K(x) |\nabla u|^2 + \frac{1}{2\pi\varepsilon^4} g_{gv}^4 \left( 1 - |u|^2 \right)^2$$

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**Sketch of the proof**

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Dimonte (SISSA)  
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Main Result

Critical velocity (Correggi, D, [CD15])

Let $\Omega_c$ be defined as the supremum of the solution of

$$\Omega_0 = \frac{4}{s+2} \left[ \mu^{gv} - \frac{1}{2\pi} g^{2}_{gv}(0) \right]$$

where $\mu^{gv}$ is defined through

$$-\frac{1}{2} g''^{gv} + \frac{1}{2} \Omega_0^2 (s+2) y^2 g^{2}_{gv} + \frac{1}{\pi} g^3_{gv} = \mu^{gv} g_{gv};$$

then if $\Omega_0 \geq \Omega_c$ then

$$E_{\Omega}^{GP} = \frac{E^{gv}}{\varepsilon^4} + \mathcal{O}(1)$$
It is possible to prove that there is a solution to the equation that defines $\Omega_c$ by considering the limits for $\Omega_0$ going both to 0 and to $+\infty$; we expect this solution to be unique.

As in [CPRY12] the main consequence of this estimate is that $\psi^{GP}_\Omega$ is strictly positive in $A_{\text{bulk}}$ and therefore the condensate has no vortices in $A_{\text{bulk}}$.

Therefore while $\Omega_0 \geq \Omega_c$ there are no vortices in the annulus so the phase transition already happened and this means that $\Omega_{c_3} \leq \frac{\Omega_c}{\varepsilon^4}$.

It is still an open problem to show that the value we just found is in fact the real critical speed for the condensate, that is that in fact $\Omega_{c_3} = \frac{\Omega_c}{\varepsilon^4}$, and to prove this one should show that for any $\Omega < \frac{\Omega_c}{\varepsilon^4}$ there are vortices inside the annulus.
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Thanks for the attention!

Non-Existence of Vortices in the Small Density Region of a Condensate (2011)

[IM06] R. Ignat, V. Millot
Energy Expansion and Vortex Location for a Two Dimensional Rotating Bose-Einstein Condensate (2006)

[CY08] M. Correggi, J. Yngvason
Energy and Vorticity in Fast Rotating Bose-Einstein Condensates (2008)

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