Phase transitions for a rotating BEC: the third critical speed

Mathematical Challenges in Quantum Mechanics 2018
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this presentation available on daniele.dimonte.it
based on a joint work with Michele Correggi
1. Mathematical Framework
   - Model for a rotating BEC: The Gross-Pitaevskii model

2. Increasing the Rotational Speed
   - Slow Rotation
   - Fast Rotation
   - Ultrarapid Rotation
   - Giant Vortex state

3. The third critical speed
   - An explicit value
   - Conclusion and perspectives
In the case of an $N$-body particle system, when $N \to +\infty$, if we assume condensation the state of the system can be described minimizing the following Gross-Pitaevskii energy functional:

$$E^{\text{phys}}_\omega[\psi] = \int_{\mathbb{R}^2} \text{d}r \left\{ \frac{1}{2} |\nabla \psi|^2 - \omega \psi^* L_z \psi + \frac{1}{\varepsilon^2} V(r)|\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\}$$

$$E^{\text{phys}}_\omega = \inf_{\|\psi\|_2^2=1} E^{\text{phys}}_\omega[\psi] = E^{\text{phys}}_\omega[\psi^{\text{phys}}]$$

- $V(r) = r^s$, $s > 2$ is the trapping potential.
- $L_z = -i (r \times \nabla)_z = -i \partial_\theta$ angular momentum.
- $\omega$ is the rotational speed of the condensate.
- $\varepsilon^{-2}$ in front of the quartic term is proportional to the 2-body scattering length of the interacting potential.
- Asymptotic $\varepsilon \to 0$ (corresponding to the Thomas-Fermi regime).
- Asymptotic for $\omega$ as $\varepsilon$ goes to 0.
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$$E_{\omega}^{\text{phys}} = \inf \frac{E_{\omega}^{\text{phys}}[\Psi]}{\|\Psi\|_2^2 = 1}$$

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Increasing the rotational speed $\omega$ we can observe different behaviors:

- formation of quantized vortices in the condensate (related to superfluidity properties): change of the phase of $\Psi_\omega^{\text{phys}}$
- different shapes of the condensate: change of the modulus of $\Psi_\omega^{\text{phys}}$

$\psi$ has a vortex in $x_0$ if around the point

$$\psi(x) \simeq e^{in\theta(x)} f(|x - x_0|)$$

with $f(0) = 0$

Numerics by K. Kasamatsu, M. Tsubota, M. Ueda
**First regime:** $\omega \ll \varepsilon^{-1}$

- When $\omega = 0$ the minimizer is radial and its support is concentrated in an area around a ball centered in zero, and we can only see the effect of the trapping potential.
- For small rotations $\omega < \omega_{c_1} = \omega_1 |\log \varepsilon|$ we can observe no effect on the state of the system; in particular $E_{\omega}^{\text{phys}} = E_0^{\text{phys}}$ and $\Psi_{\omega}^{\text{phys}} = \Psi_0^{\text{phys}}$ (Aftalion, Jerrard, Royo-Letelier, [AJR11]).
- As soon as $\omega \geq \omega_{c_1}$ vortices start to appear in the bulk of the condensate (Ignat, Millot, [IM06]) and when $\omega \gg |\log \varepsilon|$ the vortices are distributed uniformly (Correggi, Yngvason, [CY08]).

Experiments from the **Cornell Group**, Jila research center
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Increasing the Rotational Speed

Fast Rotation

**Second regime:** \( \omega \sim \varepsilon^{-1} \)

- While \( \omega \lesssim \omega_{c2} = \omega_2 \varepsilon^{-1} \) the condensate is still contained in a ball around zero and has vortices uniformly distributed in its bulk.
- When \( \omega = \omega_0 \varepsilon^{-1} \) the centrifugal force comes into play and the \( \Psi^\text{phys}_\omega \) becomes exponentially small in \( \varepsilon \) in a region close to the origin (Correggi, Pinsker, Rougerie, Yngvason, [CPRY12]).

Numerics from Fetter, Jackson, Stringari [FJS05]
Increasing the Rotational Speed

Fast Rotation

\[ E_{\omega}^{\text{phys}}[\Psi] = \int_{\mathbb{R}^2} d\mathbf{r} \left\{ \frac{1}{2} |(\nabla - i \mathbf{A}_{\text{rot}}) \Psi|^2 + \frac{1}{\varepsilon^2} \left[ r^s - \frac{1}{2} \varepsilon^2 \omega^2 r^2 + |\Psi|^2 \right] |\Psi|^2 \right\} \]

\[ \mathbf{A}_{\text{rot}} = \omega \mathbf{r}^\perp = \omega (-r_2, r_1) \]

When \( \omega \gg \varepsilon^{-1} \) the condensate gets concentrated on a thin annulus of mean radius equal to the minimum point of \( r^s - \frac{1}{2} \varepsilon^2 \omega^2 r^2 \) (notice that \( s > 2 \) is crucial here)

Rescaling the lengths by this radius we obtain

\[ E_{\Omega}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} d\mathbf{x} \left\{ \frac{1}{2} |(\nabla - i \mathbf{A}_\Omega) \psi|^2 + \Omega^2 W(\mathbf{x}) |\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\} \]

\[ \mathbf{A}_\Omega = \Omega \mathbf{x}^\perp, \quad W(\mathbf{x}) = \frac{x^s - 1}{s} - \frac{x^2 - 1}{2}, \]

\[ E_{\Omega}^{\text{GP}} = \inf_{\|\psi\|_2^2 = 1} E_{\Omega}^{\text{GP}}[\psi] = E_{\Omega}^{\text{GP}}[\psi_{\Omega}^{\text{GP}}] \]
Increasing the Rotational Speed

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\[ \mathcal{E}_{\Omega}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \mathrm{d} x \left\{ \frac{1}{2} |(\nabla - iA_{\Omega}) \psi|^2 + \Omega^2 W(x)|\psi|^2 + \frac{1}{\varepsilon^2} |\psi|^4 \right\} \]

- The potential \( W \) is positive and has one only minimum in \( x = 1, \ W(1) = 0 \)
- \( \Omega \) is the rescaled rotational speed
- When \( \omega \sim \varepsilon^{-1} \) also \( \Omega \sim \varepsilon^{-1} \)
Third regime: $\Omega \gg \varepsilon^{-1}$

- When $\Omega_c^2 \leq \Omega \ll \varepsilon^{-4}$ it was proven in [CPRY12] that $E_{\Omega}^{GP} = E_{\Omega}^{TF} + \mathcal{O}(\Omega \log(\varepsilon^4\Omega))$, where $E_{\Omega}^{TF}$ is the ground state of

$$E_{\Omega}^{TF}[\rho] = \int_{\mathbb{R}^2} dx \left[ \frac{1}{\varepsilon^2} \rho + \Omega^2 W(x) \right] \rho$$

- Using the energy asymptotic it was possible to show also that the profile of the minimizer is exponentially small in $\varepsilon$ outside a ring of radius 1 and of width $(\varepsilon\Omega)^{-\frac{2}{3}} = o(1)$

- Moreover, using the same asymptotic it is also possible to prove that the distribution of vorticity is uniform for $\Omega_c^2 \leq \Omega \ll \varepsilon^{-4}$
Increasing the Rotational Speed

- When $\Omega = \Omega_0 \varepsilon^{-4}$ the size of a single vortex becomes comparable to the width of the annulus where $\psi_{\Omega}^{GP}$ is essentially supported.
- Another transition occurs and vortices are expelled from the bulk of the condensate (Giant Vortex state).

Theorem (Correggi, Pinsker, Rougerie, Yngvason, [CPRY12])

If $\Omega_0$ is big enough then $\psi_{\Omega}^{GP}$ has no zeroes in the annulus, and therefore there are no vortices in the annulus; more precisely

$$|\psi_{\Omega}^{GP}(x)| = \frac{1}{\sqrt{2\pi x}} g_{gv}(x) (1 + o(1)),$$

$g_{gv} > 0$
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Numerics from Fetter, Jackson, Stringari, [FJS]
An essential tool in [CPRY12] was proving the approximation of the energy in terms of a Giant Vortex energy; assuming $\Omega_0 \gg 1$

$$\mathcal{E}^{gv}[g] = \int_{\mathbb{R}} dy \left\{ \frac{1}{2} |\nabla g| + \frac{1}{2} \Omega_0^2 (s + 2)y^2 g^2 + \frac{1}{2\pi} g^4 \right\}$$

$$E_{\Omega}^{GP} = \frac{E^{gv}}{\varepsilon^4} + O \left( |\log \varepsilon|^{\frac{9}{2}} \right)$$

$$E^{gv} = \inf_{\|g\|_2^2 = 1} \mathcal{E}^{gv}[g] = \mathcal{E}^{gv}[g_{gv}]$$
In particular the minimizer $g_{gv}$ is strictly positive in the annulus $A$ where $\psi_{\Omega}^{GP}$ is concentrated, so we can define $u$ such that

$$\psi_{\Omega}^{GP}(x) = \frac{1}{\sqrt{2\pi\varepsilon}} g_{gv}(x) u(x) e^{i\frac{\Omega}{2}\theta},$$

and in this case

$$\frac{E^{gv}}{\varepsilon^4} \geq E_{\Omega}^{GP} \geq \frac{E^{gv}}{\varepsilon^4} + \frac{1}{2\pi\varepsilon^2} \int_A d\mathbf{x} \ K(x) |\nabla u|^2 + \mathcal{O}\left(\log \frac{1}{\varepsilon}\right)$$

and the lower bound of this estimate reduces to estimate the positivity of the cost function; using the fact that for $\Omega_0$ large enough the profile of $g_{gv}$ is approximately gaussian, they were able to prove

$$K(x) \geq C \left(1 + \mathcal{O}\left(\Omega_0^{-\frac{1}{4}}\right)\right) g_{gv}^2(x)$$

In [CPRY12] no explicit value for the transition speed was given, due to the crucial hypotheses $\Omega_0 \gg 1$.
In particular the minimizer $g_{gv}$ is strictly positive in the annulus $A$ where $\psi_{\Omega}^{GP}$ is concentrated, so we can define $u$ such that $\psi_{\Omega}^{GP}(x) = \frac{1}{\sqrt{2\pi\varepsilon}} g_{gv}(x) u(x) e^{i[\Omega] \theta}$, and in this case

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Main Result

Critical velocity (Correggi, D, [CD16])

Let $\Omega_c$ be defined as the supremum of the solutions of

$$
\Omega_0 = \frac{4}{s+2} \left[ \mu_{gv} - \frac{1}{2\pi} g_{gv}^2(0) \right]
$$

where $\mu_{gv}$ is defined through $-\frac{1}{2} g_{gv}'' + \frac{1}{2} \Omega_0^2 (s+2) y^2 g_{gv} + \frac{1}{\pi} g_{gv}^3 = \mu_{gv} g_{gv};$

with these definitions in mind if $\Omega_0 \geq \Omega_c$ then

$$
E_{\Omega}^{GP} = \frac{E_{gv}}{\varepsilon^4} + \mathcal{O}(1)
$$

This value for the critical speed is extracted proving the positivity of $K$ and in particular that $K(x) > 0$ in $A$ if and only if $K(0) > 0$
It is possible to prove that there is a solution to the equation that defines $\Omega_c$ by considering the limits for $\Omega_0$ going both to 0 and to $+\infty$; we expect this solution to be unique.

As in [CPRY12] the main consequence of this estimate is that $\psi^{GP}_\Omega$ is strictly positive in $A$ and therefore the condensate has no vortices in $A$.

Therefore while $\Omega_0 \geq \Omega_c$ there are no vortices in the annulus so the phase transition already happened and this means that $\Omega_{c_3} \leq \frac{\Omega_c}{e^4}$.

It is still an open problem to show that the value we just found is in fact the real critical speed for the condensate, that is that in fact $\Omega_{c_3} = \frac{\Omega_c}{e^4}$, and to prove this one should show that for any $\Omega < \frac{\Omega_c}{e^4}$ there are vortices inside the annulus.
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Thanks for the attention!

Non-Existence of Vortices in the Small Density Region of a Condensate (2011)

[IM06] R. Ignat, V. Millot
Energy Expansion and Vortex Location for a Two Dimensional Rotating Bose-Einstein Condensate (2006)

[CY08] M. Correggi, J. Yngvason
Energy and Vorticity in Fast Rotating Bose-Einstein Condensates (2008)

Critical Rotational Speeds for Superfluids in Homogeneous Traps (2012)

[FJS05] A.L. Fetter, N. Jackson, S. Stringari
Rapid Rotation of a Bose-Einstein Condensate in a Harmonic Plus Quartic Trap (2005)

[CD16] M. Correggi, D. Dimonte
On the third Critical Speed for Rotating Bose-Einstein Condensates (2016)