Effective Dynamics of Bose-Einstein Condensates in the Thomas-Fermi Limit

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based on a joint work with **Michele Correggi**, **David Mitrouskas** and **Peter Pickl**

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The **Gross-Pitaevskii (GP) Theory** is an effective theory for interacting bosons at low temperature; let $\beta \in (0,1]$ and consider initially the following many-body Hamiltonian:

$$\sum_{j=1}^{N} (-\Delta_j + V(x_j)) + \sum_{1 \le j < k \le N} N^{3\beta - 1} v\left(N^{\beta}(x_j - x_k)\right)$$

If we evaluate the energy per particle of a completely factorized bosonic state of the form

$$\Psi_N(x_1, x_2, \ldots, x_N) = \psi(x_1)\psi(x_2)\ldots\psi(x_N)$$

with $\|\psi\| = 1$ one could expect as $N \to +\infty$

$$\begin{aligned}\frac{\langle \Psi_N, H_N \Psi_N \rangle}{N} &\cong \langle \psi, (-\Delta + V(r))\,\psi\rangle + \frac{1}{2}\left\langle \psi, \left(N^{3\beta} v(N^{\beta}\cdot) \star |\psi|^2\right)\psi \right\rangle \\ &\cong \langle \psi, (-\Delta + V(r))\,\psi\rangle + \frac{a}{2}\|\psi\|_4^4 =: \mathcal{E}^{\mathrm{GP}}[\psi]\end{aligned}$$

with $a = \int_{\mathbb{R}^3} v(x)dx > 0$.

- The infimum of the many-body energy is indeed well approximated by minimizing a one particle energy functional, the GP energy functional $\mathcal{E}^{\mathrm{GP}}$ (proven in [LSY00]).
- Moreover if the initial datum is factorized then (see e.g. [BOS14], [P15] and [AFP16])

$$\lim_{N\to+\infty}\left\|\gamma^{\Psi_{N,t}}-|\psi_t\rangle\langle\psi_t|\right\|=0$$

where $\gamma^{\Psi_{N,t}}$ is the one-body reduced density matrix of $\Psi_{N,t}$, with $i\partial_t\Psi_{N,t}=H_N\Psi_{N,t}$ and with ψ_t solving the time dependent GP equation

$$i\partial_t\psi_t=\left(-\Delta+V(r)\right)\psi_t+a|\psi_t|^2\psi_t.$$

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- The framework studied in the previously mentioned results was such that if we denoted as g_N the scattering length of the potential $N^{3\beta-1}v(N^{\beta}\cdot)$ then as

$$Ng_N \xrightarrow[N\to+\infty]{} a.$$

When $\beta = 1$ this is called GP limit, while if $\beta < 1$ we refer to it as GP-like limit. It is a particular type of dilute limit, meaning that if $\overline{\rho}$ represents the **mean density** then $\overline{\rho}g_N^3 \ll 1$ (indeed in this case $\overline{\rho}g_N^3 \sim N^{-2}$).

- A different framework that can be considered is the Thomas-Fermi (TF) limit, in which the gas is still assumed to be **dilute** (i.e. $\overline{\rho}g_N^3 \ll 1$) but

$$Ng_N \xrightarrow[N\to+\infty]{} +\infty.$$

This is the setting in which several experiments are carried on and it is particularly useful for **experiments on rotating Bose-Einstein Condensates**.

To study the TF limit we start from a Many-Body Hamiltonian of the form

$$H_N := \sum_{j=1}^{N} (-\Delta_j + V(x_j)) + \frac{R_N}{N} \sum_{1 \le j < k \le N} v_N(x_j - x_k)$$

where

- $V(x) = k|x|^s$ is the trapping potential, $s \geq 2$
- v_N is the interaction potential, $v_N(x) = N^{3\beta} v(N^\beta x)$; we denote $g_N = \frac{R_N}{N} \int_{\mathbb{R}^3} v_N(x) dx = \frac{R_N a}{N}$
- **Gross-Pitaevskii limit**: $N g_N = const$ as $N \to +\infty$
- **Thomas-Fermi limit**: $N g_N = R_N a \gg 1$ as $N \to +\infty$ where as before $a > 0$

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Formally the expected limiting equation is now the time-dependent GP equation with N-**dependent nonlinearity**, i.e.

$$i\partial_t\psi_t = (-\Delta + V(x))\,\psi_t + R_N|\psi_t|^2\psi_t.$$

In this case the kinetic term is negligible with respect to the other terms as $N \to +\infty$, and it was proven in [BCPY07]

$$\inf_{\|\psi\|_2=1}\left\langle \psi, \left(-\Delta + V + \frac{R_N}{2}|\psi|^2\right)\psi\right\rangle = \\ = R_N{}^{\frac{s}{s+3}}\left(E^{\mathrm{TF}} + \mathcal{O}(R_N{}^{-\frac{s+2}{2(s+3)}}\log R_N)\right)$$

$$E^{\mathrm{TF}} = \inf_{\|\psi\|_2=1}\left\langle \psi, \left(V + \frac{1}{2}|\psi|^2\right)\psi\right\rangle \propto 1$$

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$$E^{\mathrm{TF}} = \inf_{\|\rho\|_1=1, \rho\geq 0} \int dx \, \left(V(x) + \frac{1}{2}\rho(x) \right) \rho(x) \propto 1$$

To prove the previous estimates the explicit form of the trapping potential was exploited to use some interesting scaling property of the TF and GP energy. Indeed if $\rho_\lambda(x) = \lambda^3 \rho(\lambda x)$ we get

$$\mathcal{E}^{\mathrm{TF}}_g[\rho] := \int_{\mathbb{R}^3} dx \ \left(V(x) + \frac{g}{2}\rho(x) \right) \rho(x), \ \ E^{\mathrm{TF}}_g := \inf_{\|\rho\|_1 = 1, \rho \geq 0} \mathcal{E}^{\mathrm{TF}}_g[\rho]$$

$$\mathcal{E}^{\mathrm{TF}}_g[\rho_\lambda] = \lambda^{-s} \mathcal{E}^{\mathrm{TF}}_{\lambda^{s+3} g}[\rho] \Rightarrow E^{\mathrm{TF}}_{R_N} = R_N^{\frac{s}{s+3}} E^{\mathrm{TF}}_1$$

and if ρ^{TF}_g denote the corresponding minimizer we get

$$\boxed{\rho^{\mathrm{TF}}_{R_N}(x) = R_N^{-\frac{3}{s+3}} \rho^{\mathrm{TF}}_1 (R_N^{-\frac{1}{s+3}} x) \Rightarrow \left\| \rho^{\mathrm{TF}}_{R_N} \right\|_\infty = o(1), \ \left\| \rho^{\mathrm{TF}}_1 \right\|_\infty \propto 1.}$$

For GP a similar reasoning can be made. Indeed

$$\mathcal{E}^{\mathrm{GP}}_{R_N}[\psi] := \int_{\mathbb{R}^3} dx \, \left\{ |\nabla\psi(x)|^2 + V(x)\,|\psi(x)|^2 + \frac{R_N}{2}\,|\psi(x)|^4 \right\},$$
$$E^{\mathrm{GP}}_{R_N} := \inf_{\|\psi\|_2=1} \mathcal{E}^{\mathrm{GP}}_{R_N}[\psi]$$

Calling now $\varepsilon = R_N^{-\frac{s+2}{2(s+3)}}$ (notice $\varepsilon \to 0$) we get

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^3} dx \, \left\{ \varepsilon^2\,|\nabla\psi(x)|^2 + V(x)\,|\psi(x)|^2 + \frac{1}{2}\,|\psi(x)|^4 \right\},$$
$$E^{\mathrm{GP}} = \inf_{\|\psi\|_2=1} \mathcal{E}^{\mathrm{GP}}[\psi],$$
$$E^{\mathrm{GP}}_{R_N} = \varepsilon^{-\frac{2s}{s+2}} E^{\mathrm{GP}}, \ \left\|\psi^{\mathrm{GP}}_{R_N}\right\|_\infty = o(1), \ \left\|\psi^{\mathrm{GP}}\right\|_\infty = \mathcal{O}(1)$$

A rescaling **is needed** at the many-body level to observe a nontrivial behavior.

We rescale the many-body Hamiltonian in such a way that the interaction and the trapping potential scale in N in the same way.
Let U_N denote the rescaling of all lengths such that we consider the new variable $y = R_N^{-\frac{1}{s+3}} x$, then

$$U_N H_N U_N = R_N^{\frac{s}{s+3}} \left[\sum_{j=1}^{N} \left(-\varepsilon^2 \Delta_j + V(y_j) \right) + \frac{1}{N} \sum_{1 \le j < k \le N} v_{\widetilde{N}}(y_j - y_k) \right]$$

with $\widetilde{N} := R_N^{\frac{1}{\beta(s+3)}} N \to +\infty$.

Recall $\Psi_{N,t} = e^{-itH_N}\Psi_N$; if we rescale also the time as $\tau := R_N^{\frac{s}{s+3}} t$ so that the energy which is preserved is of order 1 in N and set

$$\Phi_{N,\tau} = U_N \Psi_{N,t}$$

then $\Phi_{N,\tau}$ solves

$$\begin{cases} i\partial_\tau \Phi_{N,\tau} = \left[\sum_{j=1}^{N} \left(-\varepsilon^2 \Delta_j + V(y_j)\right) + \frac{1}{N} \sum_{1\le j<k\le N} v_{\widetilde{N}}(y_j - y_k)\right] \Phi_{N,\tau} \\ \Phi_{N,\tau}\big|_{t=0} = \Phi_{N,0} = U_N \Psi_N \end{cases}$$

Theorem [M. Correggi, DD, D. Mitrouskas, P. Pickl · work in progress]

Let $\varphi_\tau^{\mathrm{H}}$ and $\varphi_\tau^{\mathrm{GP}}$ be the solutions of

$$i\partial_\tau \varphi_\tau^{\mathrm{H}} = -\varepsilon^2 \Delta \varphi_\tau^{\mathrm{H}} + V(y)\varphi_\tau^{\mathrm{H}} + v_{\widetilde{N}} \star \left|\varphi_\tau^{\mathrm{H}}\right|^2 \varphi_\tau^{\mathrm{H}}$$
$$i\partial_\tau \varphi_\tau^{\mathrm{GP}} = -\varepsilon^2 \Delta \varphi_\tau^{\mathrm{GP}} + V(y)\varphi_\tau^{\mathrm{GP}} + a\left|\varphi_\tau^{\mathrm{GP}}\right|^2 \varphi_\tau^{\mathrm{GP}}$$

with the same initial datum φ_0; assume that $\mathcal{E}^{\mathrm{GP}}[\varphi_0] \le E^{\mathrm{TF}} + \varepsilon^2 K_\varepsilon$ then for each $\beta \in [0, 1/6)$ and $\sigma \in (0, 1-6\beta)$ there exist finite constants C, C_τ and D_τ depending only on $\left\|\varphi_\tau^{\mathrm{H}}\right\|_\infty$ and $\left\|\varphi_\tau^{\mathrm{GP}}\right\|_\infty$ such that

$$\left\|\gamma^{\Phi_{N,\tau}} - |\varphi_\tau^{\mathrm{H}}\rangle\langle\varphi_\tau^{\mathrm{H}}|\right\|_{\mathscr{L}^1} \le C_\tau K_\varepsilon^2 \varepsilon^{-\frac{6}{s+2}} N^{-(1-6\beta-\sigma)} \exp\left\{C\left\|\varphi_\tau^{\mathrm{H}}\right\|_\infty \tau\right\}$$
$$\left\|\varphi_\tau^{\mathrm{H}} - \varphi_\tau^{\mathrm{GP}}\right\|_2 \le D_\tau \sqrt{K_\varepsilon}\,\varepsilon^{\frac{1}{s+2}} N^{-\frac{\beta}{2}} \exp\left\{C\left(\left\|\varphi_\tau^{\mathrm{GP}}\right\|_\infty^2 + \left\|\varphi_\tau^{\mathrm{H}}\right\|_\infty^2\right)\tau\right\}$$

Remarks

- The study of vortices in Bose Einstein Condensates is often carried on in two dimensions. Our proof does not really depend on the dimensions so **a similar result can be obtained also in $d = 2$**, meaning that we start from a two dimensional system and derive an effective equation. Related open problems are the derivation for GP-like limit with $\beta \geq \frac{1}{6}$ or the derivation of the two dimensional GP equation as an effective model for a three dimensional system trapped by a cylindrical trap.

Remarks

- At time $t = 0$ we typically have $\|\varphi_0\|_\infty = \mathcal{O}(1)$. Given the small parameter in front of the kinetic term we expect such an estimate to be true also at later times and in that case $C_\tau = \mathcal{O}(1)$ and $D_\tau = \mathcal{O}(1)$, but this still is an open question.
- The optimal case one can consider is when φ_0 is the ground state for the energy of the system. In this case we have that $\|\varphi_0\|_\infty = \mathcal{O}(1)$ and $K_\varepsilon = \mathcal{O}(|\log \varepsilon|)$.
  The timescale one then obtain is $\tau \approx \log N$.

Remarks

- In [JS15] R. Jerrard and D. Smets proved that if K_ε is of order $\mathcal{O}(|\log \varepsilon|)$ and the initial datum φ_0 has a finite number of vortices in it then they move on a timescale of order $\boxed{\tau \sim \varepsilon^{-2} |\log \varepsilon|^{-1}}$ along the level sets of the potential (spheres in this case).
- With the two previous hypotheses on the initial datum φ_0 the time scale for which the proof still holds true would be of order $\boxed{\tau \sim \log N}$. Assuming now that $\varepsilon^{-2} |\log \varepsilon|^{-1} \ll \log N$ our results holds for times long enough to observe the motion of vortices as described in [JS15].

The proof is divided in two parts:

- approximate the many-body solution $\Phi_{N,\tau}$ with a product state built from the Hartree solution $\varphi_\tau^{\mathrm{H}}$;
- estimate the difference between $\varphi_\tau^{\mathrm{H}}$ and $\varphi_\tau^{\mathrm{GP}}$.

Many-Body to Hartree

Following [P11], for the first part of the proof the idea is to look at a quantity that measures how many particles are out of the condensate: defining

$$p := |\varphi_\tau^{\mathrm{H}}\rangle\langle\varphi_\tau^{\mathrm{H}}|, \; q := 1 - p.$$

We can then aim at a Grönwall-type estimate for

$$\alpha_t := \left\langle \Phi_{N,\tau}, \frac{1}{N}\sum_{j=1}^{N} q_j \; \Phi_{N,\tau} \right\rangle = \langle \Phi_{N,\tau}, q_1 \; \Phi_{N,\tau} \rangle$$

where the choice of the first particle does not matter thanks to the symmetry of the system.

An easy computation gives

$$\begin{aligned}\partial_\tau \alpha_\tau \leq & C \left|\langle \Phi_{N,\tau}, p_1 q_2 V_{12} p_1 p_2 \Phi_{N,\tau}\rangle\right| + \\ & + C \left|\langle \Phi_{N,\tau}, q_1 q_2 V_{12} p_1 p_2 \Phi_{N,\tau}\rangle\right| + \\ & + C \left|\langle \Phi_{N,\tau}, q_1 q_2 V_{12} p_1 q_2 \Phi_{N,\tau}\rangle\right| = \\ = & I + II + III\end{aligned}$$

where $V_{12} = v_{\widetilde{N}}(y_1 - y_2) - v_{\widetilde{N}} \star \left|\varphi_\tau^{\mathrm{H}}\right|(y_1)$.
The main term is II (the first is identically zero, while the third is subleading thanks to the presence of three q's).

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Given that by definition

$$\left\| q_1 \Phi_{N,\tau} \right\|^2 = \alpha_\tau$$

we would like to "move" one q from one side to the other. To do so we use the symmetry of the wave function to get

$$\begin{aligned} II &= \left| \langle \Phi_{N,\tau}, q_1 q_2 V_{12} p_1 p_2 \Phi_{N,\tau} \rangle \right| \leq \\ &\leq \frac{1}{N} \left\| q_1 \Phi_{N,\tau} \right\| \left\| \sum_{j=2}^{N} q_j V_{1j} p_1 p_j \Phi_{N,\tau} \right\| \leq \\ &\leq \frac{1}{N} \sqrt{\alpha_\tau} \sqrt{N^2 \langle \Phi_{N,\tau}, p_1 p_2 V_{12} q_2 q_3 V_{13} p_1 p_3 \Phi_{N,\tau} \rangle + (\dots)} \leq \\ &\leq \sqrt{\alpha_\tau \left(\alpha_\tau \left\| \sqrt{V_{12}} p_1 \right\|_{op}^4 + (\dots) \right)} \end{aligned}$$

$$\left\| \sqrt{V_{12}} p_1 \right\|_{op}^4 = \left\| v_{\widetilde{N}} \star \left| \varphi_\tau^{\mathrm{H}} \right|^2 \right\|_\infty^2 \leq \| v \|_1^2 \left\| \varphi_\tau^{\mathrm{H}} \right\|_\infty^4$$

Using the inequality above we get the desired result.

Remarks:

- while the philosophy of the proof is the same, the result exposed above makes use also of a different operator which allows us to better measure the number of particles outside of the condensate;
- if we assume less regularity on the solution (for example estimates on the L^6 norm only) this quantity now becomes

$$\left\| \sqrt{V_{12}} p_1 \right\|_{op}^4 \leq \widetilde{N}^{2\beta} \| v \|_{\frac{3}{2}}^2 \left\| \varphi_\tau^{\mathrm{H}} \right\|_6^4 \lesssim C \varepsilon^{-\frac{4}{s+2}} \left| \log \varepsilon \right|^2 N^\beta$$

and the final estimate of the time gets worse: in particular **we do not reach the time scale of vortices.**

$$\left\| \sqrt{V_{12}} p_1 \right\|_{op}^4 = \left\| v_{\widetilde{N}} \star \left| \varphi_\tau^{\mathrm{H}} \right|^2 \right\|_\infty^2 \leq \|v\|_1^2 \left\| \varphi_\tau^{\mathrm{H}} \right\|_\infty^4$$

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Hartree to Gross-Pitaevskii

For the second part we try to estimate directly the derivative of the difference, so we study

$$\partial_\tau \left\|\varphi_\tau^{\mathrm{H}} - \varphi_\tau^{\mathrm{GP}}\right\|_2^2 \leq \left|\Im \left\langle \varphi_\tau^{\mathrm{H}}, \left(v_{\widetilde{N}} \star \left|\varphi_\tau^{\mathrm{H}}\right|^2 - a \left|\varphi_\tau^{\mathrm{GP}}\right|^2\right) \varphi_\tau^{\mathrm{GP}} \right\rangle\right| \leq$$
$$\leq \left|\left\langle \varphi_\tau^{\mathrm{H}} - \varphi_\tau^{\mathrm{GP}}, v_{\widetilde{N}} \star \left(\left|\varphi_\tau^{\mathrm{H}}\right|^2 - \left|\varphi_\tau^{\mathrm{GP}}\right|^2\right) \varphi_\tau^{\mathrm{GP}} \right\rangle\right| +$$
$$+ \left|\left\langle \varphi_\tau^{\mathrm{H}}, \left(v_{\widetilde{N}} \star \left|\varphi_\tau^{\mathrm{GP}}\right|^2 - a \left|\varphi_\tau^{\mathrm{GP}}\right|^2\right) \varphi_\tau^{\mathrm{GP}} \right\rangle\right|.$$

While the first term can be easily related to the L^2 difference of the solutions, the second require a more detailed estimate for the convergence of v_N to a delta.

Notice first that

$$\left| v_{\widetilde{N}} \star \left|\varphi_\tau^{\mathrm{GP}}\right|^2 (y) - a \left|\varphi_\tau^{\mathrm{GP}}(y)\right|^2 \right| =$$

$$= \left| \int dz\ \widetilde{N}^{3\beta} v(\widetilde{N}^{\beta}(y-z)) \int_0^1 ds\ \frac{\partial}{\partial s} \left|\varphi_\tau^{\mathrm{GP}}(y - s(y-z))\right|^2 \right| \leq$$

$$\leq \int dz'\ \frac{|z'|}{\widetilde{N}^{\beta}} v(z') \int_0^1 ds\ \left|\varphi_\tau^{\mathrm{GP}}\left(y - \frac{s}{\widetilde{N}^{\beta}} z'\right)\right| \left|\nabla\varphi_\tau^{\mathrm{GP}}\left(y - \frac{s}{\widetilde{N}^{\beta}} z'\right)\right|$$

$$\left|\left\langle \varphi_\tau^{\mathrm{H}}, \left(v_{\widetilde{N}} \star \left|\varphi_\tau^{\mathrm{GP}}\right|^2 - a\left|\varphi_\tau^{\mathrm{GP}}\right|^2\right) \varphi_\tau^{\mathrm{GP}} \right\rangle\right| \leq$$

$$\leq \frac{C}{\widetilde{N}^{\beta}} \left\|\nabla\varphi_\tau^{\mathrm{GP}}\right\|_2 \left\|\varphi_\tau^{\mathrm{GP}}\right\|_\infty^2 \leq \frac{C}{\widetilde{N}^{\beta}} \left\|\varphi_\tau^{\mathrm{GP}}\right\|_\infty^2 \sqrt{K_\varepsilon},$$

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$$\left| \left\langle \varphi_\tau^{\mathrm{H}}, \left(v_{\widetilde{N}} \star \left|\varphi_\tau^{\mathrm{GP}}\right|^2 - a\left|\varphi_\tau^{\mathrm{GP}}\right|^2 \right) \varphi_\tau^{\mathrm{GP}} \right\rangle \right| \leq$$

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Thanks for the attention!

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