Mathematics of the Bose Gas in the Thomas-Fermi Regime

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Physical Setting

Study of energy of N identical bosons in a box Λ in $d = 3$ (density $\rho := N/|\Lambda|$): let $E_0(N) := \inf \sigma(H_N)$

- Thermodynamic limit: density ρ is fixed, limit of infinite volume of the energy per particle

$$\mathfrak{e}(\rho) := \lim_{N\to+\infty} \frac{E_0(N)}{N}$$

- Dilute limit: study of $\mathfrak{e}(\rho)$ as ρa^3 is small (a scattering length, effective length of the interaction), *Lee-Huang-Yang* formula

$$\mathfrak{e}(\rho) = 4\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o\left(\sqrt{\rho a^3}\right)\right)$$

Goal: understanding the behaviour of $E_0(N)$ as $\rho a^3 \ll 1$

Physical Setting

In a dilute limit (at $T = 0$) one can expect that the macroscopic ground state of the system Ψ_{GS} is described in terms of a one-particle state, i.e., Bose-Einstein Condensation (BEC)

$$H_N \Psi_{\mathrm{GS}} = E_0(N)\, \Psi_{\mathrm{GS}}$$
$$\Psi_{\mathrm{GS}} \approx \psi_{\mathrm{GS}}^{\otimes N}$$

ψ_{GS} ground state of a nonlinear effective one-particle functional

$$\mathcal{E}^{\mathrm{eff}}[\psi] := \langle \psi, h\psi \rangle + \langle \psi, \mathcal{V}_{\mathrm{eff}}(\psi) \rangle$$

with h one-particle Hamiltonian and $\mathcal{V}_{\mathrm{eff}}$ nonlinear potential

Dilute Limits

Let v_N be the (N-dependent) pair interaction

- Mean-Field (Hartree)

$$v_N(\mathbf{x}) := \frac{1}{N} v(\mathbf{x}), \qquad \mathcal{V}_{\mathrm{eff}}(\psi) = \frac{1}{2}\left(v * |\psi|^2\right)|\psi|^2$$

- Gross-Pitaevskii (GP)

$$v_N(\mathbf{x}) := N^2 v(N\mathbf{x}), \qquad \mathcal{V}_{\mathrm{eff}}(\psi) = \frac{1}{2} g |\psi|^4$$

- Intermediate regimes ($\beta \in (0,1)$)

$$v_N(\mathbf{x}) := N^{3\beta-1} v\left(N^{\beta}\mathbf{x}\right), \qquad \mathcal{V}_{\mathrm{eff}}(\psi) = \frac{1}{2}\left(\int v\right)|\psi|^4$$

In all these cases a_N the scattering length of v_N satisfies $N a_N \to \frac{1}{8\pi} g$, with g constant ($\rho a_N^3 \approx N^{-2} \ll 1$)

Dilute Limits

Let v_N be the (N-dependent) pair interaction

- Mean-Field (Hartree) ($\beta = 0$)

$$v_N(\mathbf{x}) := \frac{1}{N} v(\mathbf{x}), \qquad \mathcal{V}_{\text{eff}}(\psi) = \frac{1}{2}\left(v * |\psi|^2\right)|\psi|^2$$

- Gross-Pitaevskii (GP) ($\beta = 1$)

$$v_N(\mathbf{x}) := N^2 v(N\mathbf{x}), \qquad \mathcal{V}_{\text{eff}}(\psi) = \frac{1}{2} g |\psi|^4$$

- Intermediate regimes ($\beta \in (0,1)$)

$$v_N(\mathbf{x}) := N^{3\beta-1} v\left(N^{\beta}\mathbf{x}\right), \qquad \mathcal{V}_{\text{eff}}(\psi) = \frac{1}{2}\left(\int v\right)|\psi|^4$$

In all these cases a_N the scattering length of v_N satisfies $Na_N \to \frac{1}{8\pi} g$, with g constant ($\rho a_N^3 \approx N^{-2} \ll 1$)

Thomas-Fermi Regime

In experimental settings, in particular in considering rotating systems, $Na_N \gg 1$; this is called Thomas-Fermi regime, in analogy with the density theory for large atoms

Two kinds of trap considered:

- Trapped system in $\mathbb{R}^3$
  one-particle Hilbert space $\mathfrak{h} := L^2\left(\mathbb{R}^3\right)$,
  one-particle Hamiltonian given by $h = -\Delta + U(\mathbf{x})$, with $U(\mathbf{x}) = |\mathbf{x}|^s$, $s \geq 2$
- System in a box
  one-particle Hilbert space $\mathfrak{h} := L^2(\Lambda)$, Λ a box of side length 1
  one-particle Hamiltonian given by $h = -\Delta$ with periodic boundary conditions

Thomas-Fermi Regime

Consider the following many-body Hamiltonian

$$H_N := \sum_{j=1}^{N} h_j + g_N N^{3\beta-1} \sum_{1\leq j<k\leq N} v\left(N^{\beta}\left(\mathrm{x}_j - \mathrm{x}_k\right)\right)$$

defined on $\mathcal{H}_N := \mathfrak{h}^{\otimes_{\mathrm{s}} N}$

- Without loss of generality $\int v = 1$; then the scattering length of $g_N N^{3\beta} v\left(N^{\beta}\cdot\right)$ is given for $\beta \in [0,1)$ by

  $$Na_N = \frac{1}{8\pi} g_N \left(1 + o\left(1\right)\right)$$

  therefore we require $g_N \gg 1$ (TF regime)
- If $g_N \leq N^{\frac{2(s+3)}{3(s+2)}}$ this is still a *dilute limit* (for a system in a box this means $g_N \leq N^{2/3}$)

Mathematical Setting

For any one-particle observable A, many-body state $\Psi \in \mathcal{H}_N$

$$\langle \Psi, \sum_{j=1}^{N} A_j \Psi \rangle = N \operatorname{tr}\left[\gamma_\Psi^{(1)} A\right]$$

where $\gamma_\Psi^{(1)}$ is the 1-particle reduced density matrix

Complete BEC

Given a many-body state $\Psi \in \mathcal{H}_N$ and a one-particle state $\varphi \in \mathfrak{h}$

$$\langle \varphi, \gamma_\Psi^{(1)} \varphi \rangle = \langle \Psi, (|\varphi\rangle\langle\varphi|)_1 \Psi \rangle \xrightarrow[N\to+\infty]{} 1$$

a macroscopic fraction of the particles occupies the same one-particle state

Equivalently

$$\gamma_\Psi^{(1)} \to P_\varphi := |\varphi\rangle\langle\varphi|, \qquad \text{in } \mathfrak{S}_1(\mathfrak{h})$$

Introduction

Stationary Problem

Dynamical Problem

Conclusion

Setting

We consider a system in a box with periodic boundary conditions ($\mathfrak{h} = L^2(\Lambda)$)

$$H_N = \sum_{j=1}^{N} -\Delta_j + g_N N^{3\beta-1} \sum_{1 \le j < k \le N} v\left(N^\beta (\mathsf{x}_j - \mathsf{x}_k)\right)$$

Crucial assumption: v radial, compactly supported and of positive type ($\widehat{v} \geq 0$)

$$\begin{aligned} E_0(N) &:= \inf \sigma(H_N) \\ &= \inf \left\{ \langle \Psi, H_N \Psi \rangle : \ \Psi \in \mathfrak{h}^{\otimes_s N}, \ \|\Psi\| = 1 \right\} \end{aligned}$$

Trivial Upper Bound

Translation invariant system, first guess for an upper bound: *the constant function*

$$
\begin{aligned}
E_0(N) &\leq \langle 1, H_N 1 \rangle \\
&= \frac{g_N(N-1)}{2} \widehat{v}(\mathbf{0})
\end{aligned}
$$

We want to match this upper bound with a corresponding lower bound and estimate the number of excited particles

Number of Excited Particles

Let $a_{\mathbf{p}}$ annihilate (respectively $a_{\mathbf{p}}^*$ create) a particle of momentum $\mathbf{p}$; for bosons, they satisfy the canonical commutation relations (CCR)

$$\begin{aligned} \left[a_{\mathbf{p}}, a_{\mathbf{q}}^*\right] &= \delta_{\mathbf{p},\mathbf{q}}, \\ \left[a_{\mathbf{p}}, a_{\mathbf{q}}\right] &= \left[a_{\mathbf{p}}^*, a_{\mathbf{q}}^*\right] = 0, \end{aligned} \qquad \mathbf{p},\ \mathbf{q} \in \Lambda^* := (2\pi\mathbb{Z})^3$$

The number of excited particles is measured by

$$\begin{aligned} \mathcal{N}_+ &:= \sum_{\mathbf{p}\neq 0} a_{\mathbf{p}}^* a_{\mathbf{p}} \\ &\leq \frac{1}{(2\pi)^2} \sum_{\mathbf{p}\neq 0} |\mathbf{p}|^2 a_{\mathbf{p}}^* a_{\mathbf{p}} = \frac{1}{(2\pi)^2} \sum_{j=1}^{N} -\Delta_j \end{aligned}$$

Lower Bound

Following **[S11]** and using $\widehat{v} \geq 0$ one can estimate the potential as

$$g_N N^{3\beta-1} \sum_{1\leq j<k\leq N} v\left(N^{\beta}\left(\mathsf{x}_j - \mathsf{x}_k\right)\right) \geq \frac{g_N\left(N-1\right)}{2}\widehat{v}\left(\mathbf{0}\right) + \frac{g_N}{2}\left(\widehat{v}\left(\mathbf{0}\right) - N^{3\beta} v\left(\mathbf{0}\right)\right)$$

So if $\|\Psi\| = 1$ we get

$$(2\pi)^2 \langle \Psi, \mathcal{N}_+ \Psi\rangle \leq \langle \Psi, H_N \Psi\rangle - \frac{g_N\left(N-1\right)}{2}\widehat{v}\left(\mathbf{0}\right) + \mathcal{O}\left(g_N N^{3\beta}\right)$$

[S11] Seiringer, "The Excitation Spectrum for Weakly Interacting Bosons"

Theorem

Assume v compactly supported and of positive type ($\widehat{v} \geq 0$) and $\beta < 1/3$

$$E_0(N) = \frac{g_N(N-1)}{2}\widehat{v}(\mathbf{0}) + o(g_N N)$$

Moreover, if $g_N \ll N^{1-3\beta}$, the estimate for $\mathcal{N}_+$ guarantees *complete BEC in the ground state*, i.e., if Ψ_{GS} is the (normalized) ground state

$$1 - \langle \Psi_{\mathrm{GS}}, (|1\rangle\langle 1|)_1 \Psi_{\mathrm{GS}} \rangle \leq \mathcal{O}\left(g_N N^{3\beta-1}\right)$$

Related questions:

- $\beta \geq 1/3$: different ideas are needed
- Next-to-leading order approximation: Bogoliubov theory

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Assume v compactly supported and of positive type ($\widehat{v} \geq 0$) and $\beta < 1/3$

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Related questions:

- $\beta \geq 1/3$: different ideas are needed
- Next-to-leading order approximation: Bogoliubov theory

Bogoliubov Theory

In **[LNSS15]** a useful decomposition was introduced: let $\Psi \in \mathcal{H}_N$, $\varphi_0 \in \mathfrak{h} \setminus \{0\}$

$$\Psi = \Psi_0 \varphi_0^{\otimes N} + \Psi_1 \otimes_s \varphi_0^{\otimes (N-1)} + \ldots + \Psi_N$$

$$\Psi \to (\Psi_0, \Psi_1, \ldots, \Psi_N) =: U_N \Psi$$

U_N is a unitary map from $\mathcal{H}_N$ to the Fock space of the excitations

$$\mathcal{F}_+^{\leq N} := \bigoplus_{j=0}^{N} \mathfrak{h}_+^{\otimes_s j} \hookrightarrow \mathcal{F}_+ := \bigoplus_{j \geq 0} \mathfrak{h}_+^{\otimes_s j}, \; \mathfrak{h}_+ := \{\varphi_0\}^\perp, \; \varphi_0 := 1$$

$\mathcal{N}_+$ corresponds to the number operator on this Fock space

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[LNSS15] Lewin, Nam, Serfaty, Solovej, "Bogoliubov Spectrum of Interacting Bose Gases"

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[LNSS15] Lewin, Nam, Serfaty, Solovej, "Bogoliubov Spectrum of Interacting Bose Gases"

Bogoliubov Theory

a and a^* are not well defined on $\mathcal{F}_+^{\leq N}$ (they are defined on the bigger space $\mathcal{F}_+$):

$$b_{\mathbf{p}} := \sqrt{\frac{N-\mathcal{N}_+}{N}}\, a_{\mathbf{p}}, \;\; b^*_{\mathbf{p}} := a^*_{\mathbf{p}} \sqrt{\frac{N-\mathcal{N}_+}{N}}, \;\; \mathbf{p} \in \Lambda^*_+ := \Lambda^* \setminus \{\mathbf{0}\}$$

b and b^* *almost* satisfy CCRs

$$\begin{aligned} \left[b_{\mathbf{p}}, b^*_{\mathbf{q}}\right] &= \frac{N-\mathcal{N}_+}{N}\delta_{\mathbf{pq}} - \frac{1}{N} a^*_{\mathbf{q}} a_{\mathbf{p}}, \\ \left[b_{\mathbf{p}}, b_{\mathbf{q}}\right] &= \left[b^*_{\mathbf{p}}, b^*_{\mathbf{q}}\right] = 0, \end{aligned} \qquad \mathbf{p},\; \mathbf{q} \in \Lambda^*_+$$

Bogoliubov Theory

$$U_N \Psi = (\Psi_0, \Psi_1, \ldots, \Psi_N)$$

We define $\mathcal{L}_N := U_N H_N U_N^*$

$$\begin{aligned} \mathcal{L}_N =& \frac{g_N (N-1)}{2} \widehat{v}(\mathbf{0}) + \mathbb{H}_N + \mathcal{R}_N, \\ \mathbb{H}_N :=& \sum_{\mathbf{p} \neq 0} \left[|\mathbf{p}|^2 b_{\mathbf{p}}^* b_{\mathbf{p}} + \frac{g_N}{2} \widehat{v}\left(\frac{\mathbf{p}}{N^\beta}\right) \left(2 b_{\mathbf{p}}^* b_{\mathbf{p}} + b_{\mathbf{p}} b_{-\mathbf{p}} + b_{\mathbf{p}}^* b_{-\mathbf{p}}^*\right) \right] \end{aligned}$$

$\mathcal{R}_N$ contains higher order terms (cubic and quartic terms) that we shall now drop but that require further analysis

Bogoliubov Theory

$\mathbb{H}_N$ is *almost* diagonalizable:

$$d_{\mathbf{p}} := \frac{b_{\mathbf{p}} + \alpha_{\mathbf{p}} b^*_{-\mathbf{p}}}{\sqrt{1-\alpha_{\mathbf{p}}^2}}, \qquad d^*_{\mathbf{p}} := \frac{b^*_{\mathbf{p}} + \alpha_{\mathbf{p}} b_{-\mathbf{p}}}{\sqrt{1-\alpha_{\mathbf{p}}^2}}, \qquad \alpha_{\mathbf{p}} \in [0,1)$$

$$\mathbb{H}_N = E_{\mathrm{Bog}} + \sum_{\mathbf{p}\in\Lambda_+^*} \epsilon_{\mathbf{p}} d^*_{\mathbf{p}} d_{\mathbf{p}} + \mathcal{S}_N, \qquad \epsilon_{\mathbf{p}} \geq 0$$

$\mathcal{S}_N$ negligible with respect to E_{Bog}

Problem: d and d^* do not satisfy CCR

Lemma

Let $\nabla\widehat{v},\ \frac{\widehat{v}}{|\mathbf{p}|} \in L^1\left(\mathbb{R}^3\right)$

$$E_{\mathrm{Bog}} = -\left|E_{\mathrm{Bog}}\right| = -g_N^2 N^{\beta} \int_{\mathbb{R}^3} d\mathbf{p} \ \left(\frac{\widehat{v}\left(\mathbf{p}\right)}{2\left|\mathbf{p}\right|}\right)^2 + o\left(g_N^2 N^{\beta}\right)$$

Guess for Next Order

If $\beta < \frac{1}{3}$

$$E_0\left(N\right) = \frac{g_N\left(N-1\right)}{2}\widehat{v}\left(\mathbf{0}\right) - g_N^2 N^{\beta} \int_{\mathbb{R}^3} d\mathbf{p} \ \left(\frac{\widehat{v}\left(\mathbf{p}\right)}{2\left|\mathbf{p}\right|}\right)^2 \left(1 + o\left(1\right)\right)$$

Perspectives

$$\mathcal{L}_N = \frac{g_N(N-1)}{2}\widehat{v}(\mathbf{0}) + E_{\mathrm{Bog}} + \sum_{\mathbf{p}\in\Lambda_+^*} \epsilon_{\mathbf{p}} d_{\mathbf{p}}^* d_{\mathbf{p}} + \mathcal{R}_N + \mathcal{S}_N$$

The tools of **[BBCS19]** could be useful in studying $\mathcal{L}_N$

- Estimate $\mathcal{R}_N$ and $\mathcal{S}_N$ in terms of $\mathcal{N}_+$ to show that those are subleading when considering the ground state energy
- Use a suitable *Bogoliubov transformation* $\mathcal{V}_N$ so that $\mathcal{V}_N d_{\mathbf{p}}^* d_{\mathbf{p}} \mathcal{V}_N^* \approx a_{\mathbf{p}}^* a_{\mathbf{p}}$
- $\mathcal{V}_N$ would also be useful to study the spectrum of excitations of H_N in terms of the values $\epsilon_{\mathbf{p}}$

[BBCS19] Boccato, Brennecke, Cenatiempo, Schlein, "Optimal Rate for Bose–Einstein Condensation in the Gross–Pitaevskii Regime"

Setting

We consider a trapped system in $\mathbb{R}^3$ $(\mathfrak{h} = L^2(\mathbb{R}^3))$

$$H_N := \sum_{j=1}^{N} \left(-\Delta_j + U(\mathbf{x}_j)\right) + g_N N^{3\beta-1} \sum_{1\leq j<k\leq N} v\left(N^\beta (\mathbf{x}_j - \mathbf{x}_k)\right)$$

with $U(\mathbf{x}) := |\mathbf{x}|^s$, $s \geq 2$

$$\begin{cases} i\partial_t \Psi_{N,t} = H_N \Psi_{N,t} \\ \Psi_{N,t}\big|_{t=0} = \Psi_{N,0} \end{cases}$$

Goal: understand whether complete BEC is preserved by time evolution, i.e.

$$\gamma^{(1)}_{\Psi_{N,0}} \to P_{\psi_0} \text{ in } \mathfrak{S}_1(\mathfrak{h}) \Longrightarrow \gamma^{(1)}_{\Psi_{N,t}} \to P_{\psi_t} \text{ in } \mathfrak{S}_1(\mathfrak{h})$$

Gross-Pitaevskii Equation

Expected limiting equation: the time-dependent GP equation

$$\begin{cases} i\partial_t \psi_t = \left(-\Delta + U(\mathbf{x})\right)\psi_t + g_N |\psi_t|^2 \psi_t \\ \psi_t|_{t=0} = \psi_0 \end{cases}$$

Idea: in the ground state energy the kinetic term is negligible; applying the unitary transformation $\psi(\mathbf{x}) \to \lambda^{3/2}\psi(\lambda\mathbf{x})$ we get

$$\inf_{\|\psi\|_2=1} \langle \psi, \left(-\Delta + U + \frac{g_N}{2}|\psi|^2\right)\psi\rangle$$

$$= g_N^{\frac{s}{s+3}} \inf_{\|\phi\|_2=1} \langle \phi, \left(-g_N^{-\frac{s+2}{s+3}}\Delta + U + \frac{1}{2}|\phi|^2\right)\phi\rangle$$

$$=: \varepsilon^{-\frac{2s}{s+2}} \inf_{\|\phi\|_2=1} \mathcal{E}^{\mathrm{GP}}[\phi]$$

calling $\varepsilon := g_N^{-(s+2)/2(s+3)}$

Gross-Pitaevskii Equation

Expected limiting equation: the time-dependent GP equation

$$\begin{cases} i\partial_t \psi_t = (-\Delta + U(\mathbf{x}))\,\psi_t + g_N |\psi_t|^2 \psi_t \\ \psi_t\big|_{t=0} = \psi_0 \end{cases}$$

Idea: in the ground state energy the kinetic term is negligible; applying the unitary transformation $\psi(\mathbf{x}) \to \lambda^{3/2}\psi(\lambda \mathbf{x})$ we get

$$\begin{aligned} &\inf_{\|\psi\|_2=1} \langle \psi, \left(-\Delta + U + \frac{g_N}{2}|\psi|^2\right)\psi\rangle \\ &\quad = \varepsilon^{-\frac{2s}{s+2}} \inf_{\|\phi\|_2=1} \langle \phi, \left(-\varepsilon^2\Delta + U + \frac{1}{2}|\phi|^2\right)\phi\rangle \\ &\quad =: \varepsilon^{-\frac{2s}{s+2}} \inf_{\|\phi\|_2=1} \mathcal{E}^{\mathrm{GP}}[\phi] \end{aligned}$$

calling $\varepsilon := g_N^{-(s+2)/2(s+3)}$

Thomas-Fermi Energy

Dropping the kinetic term we obtain the TF energy functional

$$\mathcal{E}^{\mathrm{TF}}[\rho] := \int_{\mathbb{R}^3} d\mathsf{x} \, \left(U(\mathsf{x}) + \frac{1}{2}\rho(\mathsf{x}) \right) \rho(\mathsf{x}),$$

$$E^{\mathrm{TF}} := \inf_{\|\rho\|_1 = 1, \ \rho \geq 0} \mathcal{E}^{\mathrm{TF}}[\rho],$$

$$\mathcal{E}^{\mathrm{GP}}[\phi] = \int_{\mathbb{R}^3} d\mathsf{x} \, \left(\varepsilon^2 |\nabla\psi(\mathsf{x})|^2 U(\mathsf{x}) + \frac{1}{2}\rho(\mathsf{x}) \right) \rho(\mathsf{x})$$

$$E^{\mathrm{GP}} := \inf_{\|\phi\|_2 = 1} \mathcal{E}^{\mathrm{GP}}[\phi] = E^{\mathrm{TF}} + \mathcal{O}(\varepsilon |\log \varepsilon|) \ \ (\textbf{[BCPY07]})$$

A rescaling *is needed* both at the many-body level and at the one-particle level to observe a nontrivial behavior

[BCPY07] Bru, Correggi, Pickl, Yngvason, "The TF Limit for Rapidly Rotating Bose Gases in Anharmonic Traps"

Rescaling of the Hamiltonian

The corresponding rescaling for the many-body system is

$$\Phi_{N,\tau}(\mathbf{y}_1,\ldots,\mathbf{y}_N) := \varepsilon^{-\frac{3N}{s+2}} \Psi_{N,t}(\mathbf{x}_1,\ldots,\mathbf{x}_N)$$
$$\tau := \varepsilon^{-\frac{2(s+3)}{s+2}} t, \qquad \mathbf{y} := \varepsilon^{-\frac{2}{s+2}} \mathbf{x}$$

$\Phi_{N,\tau}$ solves

$$\begin{cases} i\partial_\tau \Phi_{N,\tau} = K_N \Phi_{N,\tau} \\ \Phi_{N,\tau}\big|_{t=0} = \Phi_{N,0} \equiv \Phi_N \end{cases}$$

$$K_N := \sum_{j=1}^{N} \left(-\varepsilon^2 \Delta_j + U(\mathbf{y}_j)\right) + N^{-1} \widetilde{N}^{3\beta} \sum_{1 \le j < k \le N} v\left(\widetilde{N}^{\beta}(\mathbf{y}_j - \mathbf{y}_k)\right)$$

with $\widetilde{N} := \varepsilon^{-\frac{2}{\beta(s+2)}} N \gg N$

Rescaling the Limiting Equation

A similar rescaling for the GP equation gives

$$\phi_\tau(\mathbf{y}) = \varepsilon^{-\frac{3}{s+2}} \psi_t(\mathbf{x})$$

ϕ_τ solves the time-dependent rescaled GP equation

$$\begin{cases} i\partial_\tau \phi_\tau = -\varepsilon^2 \Delta \phi_\tau + U(y)\phi_\tau + |\phi_\tau|^2 \phi_\tau \\ \phi_\tau|_{\tau=0} = \phi_0 \end{cases}$$

We also use an intermediate equation, the time-dependent Hartree (H) equation

$$\begin{cases} i\partial_\tau \varphi_\tau = -\varepsilon^2 \Delta \varphi_\tau + U(y)\varphi_\tau + v_{\widetilde{N}} * |\varphi_\tau|^2 \varphi_\tau \\ \varphi|_{\tau=0} = \phi_0 \end{cases}$$

where $v_{\widetilde{N}}(\mathbf{x}) := \widetilde{N}^{3\beta} v\left(\widetilde{N}^\beta \mathbf{x}\right)$

Conjecture

Let ϕ_0 be the initial datum of the GP equation

$$\|\phi_0\|_\infty = \mathcal{O}(1) \Longrightarrow \sup_{\tau\in[0,+\infty)} \|\phi_\tau\|_\infty \leq \mathcal{O}(1)$$

Theorem

Assume that the Conjecture holds true and

$$\left\|\gamma^{(1)}_{\Psi_{N,0}} - P_{\psi_0}\right\|_{\mathfrak{S}^1} \leq \zeta_N \ll 1$$

$$\mathcal{E}^{\mathrm{GP}}[\phi_0] - E^{\mathrm{GP}} \leq \xi_N$$

$$\varepsilon \gg [(1-6\beta-\delta)\log N]^{-\frac{s+2}{2(s+3)}}$$

then for each $\beta \in [0, 1/6)$ there is *complete BEC on* ψ_t, i.e.

$$\left\|\gamma^{(1)}_{\Psi_{N,t}} - P_{\psi_t}\right\|_{\mathfrak{S}^1} \ll 1$$

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Let ϕ_0 be the initial datum of the GP equation

$$\|\phi_0\|_\infty = \mathcal{O}(1) \Longrightarrow \sup_{\tau\in[0,+\infty)} \|\phi_\tau\|_\infty \leq \mathcal{O}(1)$$

Theorem

Assume that the Conjecture holds true and

$$\left\|\gamma^{(1)}_{\Psi_{N,0}} - P_{\psi_0}\right\|_{\mathfrak{S}^1} \leq \zeta_N = \varepsilon^{-6/(s+2)} N^{\beta-1}$$

$$\mathcal{E}^{\mathrm{GP}}[\phi_0] - E^{\mathrm{GP}} \leq \xi_N = |\log\varepsilon|\, \varepsilon^{-(5s+6)/2(s+2)} N^{-\beta}$$

$$\varepsilon \gg [(1-6\beta-\delta)\log N]^{-\frac{s+2}{2(s+3)}}$$

then for each $\beta \in [0, 1/6)$ there is *complete BEC on* ψ_t, i.e.

$$\left\|\gamma^{(1)}_{\Psi_{N,t}} - P_{\psi_t}\right\|_{\mathfrak{S}^1} \ll 1$$

Remarks

- Similar result is achievable also in $d = 2$
- Open question is to get over $\beta = 1/6$; also related to stationary problem limitations
- (HP1) means that there is BEC in the initial datum Ψ_0 on the state ϕ_0
- (HP2) means that the GP initial datum ϕ_0 is close to a ground state in energy: crucial to prove that the Hartree solution is close to the GP solution
- (HP3) is necessary to prove condensation on a state ψ_τ; still allows for a dilute limit

$$\left\| \gamma^{(1)}_{\Psi_{N,0}} - P_{\psi_0} \right\|_{\mathfrak{S}^1} \ll 1 \tag{HP1}$$

$$\mathcal{E}^{\mathrm{GP}}[\phi_0] - E^{\mathrm{GP}} \leq \xi_N \tag{HP2}$$

$$\varepsilon \gg \left[(1 - 6\beta - \delta) \log N\right]^{-\frac{s+2}{2(s+3)}} \tag{HP3}$$

Sketch of the proof

Two parts:

- Approximate the $\gamma^{(1)}_{\Phi_{N,\tau}}$ with P_{φ_τ}
- Estimate the difference between φ_τ and ϕ_τ

Main ingredients:

- Tools developed in **[P11]**
- Energy estimates for the one-particle problem

[P11] Pickl, "A Simple Derivation of Mean Field Limits for Quantum Systems"

Many-Body to Hartree

Similarly to **[P11]**, the goal is obtaining a Grönwall-type estimate for

$$\alpha_\tau := 1 - \langle \Phi_{N,\tau}, (|\varphi_\tau\rangle \langle\varphi_\tau|)_1 \, \Phi_{N,\tau}\rangle$$

We need to estimate terms of the form

$$\left\| v_{\widetilde{N}} * |\varphi_\tau|^2 \right\|_\infty \leq \|v\|_1 \, \|\varphi_\tau\|_\infty^2$$

Using the Conjecture we get the desired result; if we do not assume it, we can only use the kinetic energy: *we do not reach the time scale of vortices* (compare with **[JS15]**)

[JS15] Jerrard, Smets, "Vortex dynamics for the two-dimensional non-homogeneous Gross-Pitaevskii equation"

Hartree to Gross-Pitaevskii

$$\partial_\tau \|\varphi_\tau - \phi_\tau\|_2^2 \le \left|\mathrm{Im}\langle \varphi_\tau, \left(v_{\widetilde{N}} * |\varphi_\tau|^2 - a\,|\phi_\tau|^2\right)\phi_\tau\rangle\right| \le$$

$$\le \left|\langle \varphi_\tau - \phi_\tau, v_{\widetilde{N}} * \left(|\varphi_\tau|^2 - |\phi_\tau|^2\right)\phi_\tau\rangle\right| +$$

$$+ \left|\langle \varphi_\tau, \left(v_{\widetilde{N}} * |\phi_\tau|^2 - |\phi_\tau|^2\right)\phi_\tau\rangle\right|$$

To prove convergence of this last two terms use L^2 difference of the solutions for the first term and $v_N \to \delta$ as a distribution for the second one:

$$\left|\langle \varphi_\tau, \left(v_{\widetilde{N}} * |\phi_\tau|^2 - |\phi_\tau|^2\right)\phi_\tau\rangle\right| \le \frac{C}{\widetilde{N}^\beta}\,\|\nabla\phi_\tau\|_2\,\|\phi_\tau\|_\infty^2$$

Conclusion

- There is BEC in the Thomas Fermi limit, at least in a scaling with $\beta < 1/3$
  - **Q:** What happens if $\beta > 1/3$?
  - **Q:** Is Bogoliubov theory correct for $\beta < 1/3$?

- Condensation is preserved under suitable assumptions of regularity on the solution
  - **Q:** How to prove the Conjecture?
  - **Q:** Is it possible to prove the same result for larger values of β?
  - **Q:** Vortices are encoded in the vorticity measure, which depends on the gradiend of the solution; can a similar result be proven in a stronger (e.g. H^1) norm?

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So long, and thanks for all the fish!