Dynamics of a N-body Vortex State in a Bose-Einstein Condensate

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based on a joint work with **Michele Correggi** and **Peter Pickl**

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The **Gross-Pitaevskii (GP) theory** is an effective theory for interacting bosons; indeed, consider the Hamiltonian for a gas of bosons with short range interaction

$$H_N := \sum_{j=1}^{N} (-\Delta_j + V(r_j)) + \sum_{1 \le j < k \le N} N^{3\beta - 1} v\left(N^{\beta}(x_j - x_k)\right)$$

where

- $v \in C_0^\infty(\mathbb{R}^3)$, $v \geq 0$;
- $V(x) = k|x|^s$ is a trapping potential where $k > 0$ and $s \geq 2$.

If we consider the energy per particle of a completely factorized bosonic state

$$\Psi_N(x_1, x_2, \ldots, x_N) = \varphi(x_1)\varphi(x_2)\ldots\varphi(x_N)$$

with $\|\varphi\| = 1$ we get as $N \to +\infty$

$$\begin{aligned}\frac{\langle \Psi_N, H_N \Psi_N \rangle}{N} &\cong \langle \varphi, (-\Delta + V(r))\,\varphi \rangle + \frac{1}{2}\langle \varphi, \left(N^{3\beta} v(N^{\beta}\cdot) \star |\varphi|^2 \right) \varphi \rangle \\ &\cong \langle \varphi, (-\Delta + V(r))\,\varphi \rangle + \frac{a}{2}\|\varphi\|_4^4 =: \mathcal{E}[\varphi]\end{aligned}$$

with $a = \int v(x)dx > 0$.

- The infimum of the many-body energy is attained by minimizing a one particle energy functional, the GP energy functional $\mathcal{E}$ (proven in [LSY00]).
- If the initial datum is factorized, then Ψ_t solution of

$$i\partial_t \Psi_t = H_N \Psi_t$$

stays "close" in a sense to be made precise below to a tensor product of identical states φ_t, solving to the GP equation

$$i\partial_t \varphi_t = (-\Delta + V(r))\,\varphi_t + a|\varphi_t|^2 \varphi_t.$$

The result usually proven in this setting is (see in particular [BOS14] and [P15])

$$\lim_{N\to+\infty} \left\| \gamma^{\Psi_t} - |\varphi_t\rangle\langle\varphi_t| \right\| = 0$$

where γ^{Ψ_t} is the one-body reduced density matrix of Ψ_t, i.e.

$$\gamma^{\Psi} := Tr_{2,\dots,N} |\Psi\rangle\langle\Psi|.$$

Remark

Notice that in the typical setting considered for the derivation of the GP equation

$$i\partial_t \varphi_t = \left(-\Delta + V(r)\right)\varphi_t + a|\varphi_t|^2 \varphi_t$$

we have a 3 dimensional gas with the value of a kept constant.

R. L. Jerrard and D. Smets in [JS15] studied the dynamics of a vortex in the framework of the **2D Gross-Pitaevskii time-dependent equation**:

$$i\partial_t u_t = -\Delta u_t + a\left(V + |u_t|^2\right) u_t \tag{2DGP}$$

where

- the potential $V(y) = k|y|^s$, $s \geq 2$ is a trapping potential;
- the parameter $a \to +\infty$ (**Thomas-Fermi (TF) limit** of the Gross-Pitaevskii equation).

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The GP equation is known to preserve the **Gross-Pitaevskii energy**

$$\mathcal{E}^{\mathrm{GP}}[u] := \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |\nabla u| + \frac{1}{2\varepsilon^2} \left(V + \frac{1}{2}|u|^2 \right) |u|^2 \right\} dy.$$

In particular if η is a **ground state for** $\mathcal{E}^{\mathrm{GP}}$, i.e.

$$\mathcal{E}^{\mathrm{GP}}[\eta] = \inf_{\|u\|_2=1} \mathcal{E}^{\mathrm{GP}}[u] =: E^{\mathrm{GP}}$$

we can take it as the background for our vortices, meaning that we will assume that the state we consider is η times some function containing vortices:

$$u = \eta\nu \text{ with } |\nu| \simeq 1 \text{ and } \deg \nu \neq 0.$$

When $\varepsilon \to 0$ the description of the ground state for $\mathcal{E}^{\mathrm{GP}}$ can be done in term of another effective energy, called the **Thomas Fermi energy**

$$\mathcal{E}^{\mathrm{TF}}[\rho] := \int_{\mathbb{R}^2} \frac{1}{2} \left(V + \frac{1}{2}\rho \right) \rho \; dy$$
$$E^{\mathrm{TF}} := \inf_{\|\rho\|_1 = 1} \mathcal{E}^{\mathrm{TF}}[\rho] = \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}].$$

Indeed in the limit for $\varepsilon \to 0$ we get

$$E^{\mathrm{GP}} = \frac{1}{\varepsilon^2} E^{\mathrm{TF}} + \mathcal{O}\left(\frac{|\log \varepsilon|}{\varepsilon} \right)$$

and using this information one gets that

$$\eta^2 \xrightarrow[\varepsilon \to 0]{} \rho^{\mathrm{TF}} \text{ in } L^2\left(\mathbb{R}^2\right).$$

Being $\rho^{\mathrm{TF}} = (\lambda_0 - V(x))_+$ with chemical potential λ_0 determined in such a way that $\left\|\rho^{\mathrm{TF}}\right\|_1 = 1$, the idea is to choose an initial state u_0 such that

$$\mathcal{E}^{\mathrm{GP}}[u_0] \leq E^{\mathrm{GP}} + \pi\rho^{\mathrm{TF}}(\alpha_0)\left|\log\varepsilon\right| + o\left(\left|\log\varepsilon\right|\right),$$
$$\operatorname{curl} j\left(\frac{u_0}{\sqrt{\rho^{\mathrm{TF}}}}\right) \xrightarrow[\varepsilon\to 0]{} 2\pi\delta_{\alpha_0}$$

where the first inequality exploits the fact that the energy of vortices is of order $\log\varepsilon$ and the second one introduces the vorticity measure characterizing the vortex structure of u_0; indeed

$$\begin{matrix} \frac{|u_0|}{\sqrt{\rho^{\mathrm{TF}}}} = 1 \\ \deg\left(u_0, \partial B_r(\alpha_0)\right) = n \in \mathbb{Z} \end{matrix} \implies \operatorname{curl} j\left(\frac{|u_0|}{\sqrt{\rho^{\mathrm{TF}}}}\right) = 2\pi n\ \delta_{\alpha_0}.$$

If u_t is the solution to the (2DGP) with initial datum u_0, calling $u_t^\varepsilon := u_{|\log\varepsilon| t}$ they were able to prove that

$$\text{curl}\, j\left(\frac{u_t^\varepsilon}{\sqrt{\rho^{\mathrm{TF}}}}\right) \xrightarrow[\varepsilon\to 0]{} 2\pi\delta_{\alpha_t}$$

also at later times, with α_t solution of

$$\dot{\alpha}_t = \nabla^\perp \log(\rho^{\mathrm{TF}}(\alpha_t))$$

describing the motion of the vortex center.

$$\dot{\alpha}_t = \nabla^{\perp} \log(\rho^{\mathrm{TF}}(\alpha_t)) = \frac{\nabla^{\perp}\rho^{\mathrm{TF}}(\alpha_t)}{\rho^{\mathrm{TF}}(\alpha_t)}$$

- The solution α_t is such that $\rho^{\mathrm{TF}}(\alpha_t) = \mathit{const}$, and given the explicit form of ρ^{TF} it is easy to see that the vortices move along the level sets of the potential V (which actually determines ρ^{TF}).
- The complete result in [JS15] is even stronger and shows that this is true also for more than one vortex, and that the vortices motion is well described as long as their path do not intersect.
- Therefore, given the shape of the trapping potential V and the initial position of the vortices their paths can be determined.

The question that naturally arises now is whether one can describe the motion of vortices at the many-body level, in particular in the TF-limit. We consider a gas of bosons trapped by a cylindrical trap with short range interaction; this means that if $\mathbb{R}^3 \ni x = (r, z) \in \mathbb{R}^2 \times \mathbb{R}$ we consider a **radial** trapping potential $V(r)$ and an **infinite well trapping** in the z direction, meaning that our system will be defined on

$$\mathcal{H} := L^2 \left(\mathbb{R}^2 \times \left[-\frac{h}{2}, \frac{h}{2} \right] \right)$$

with $h \gg 1$ representing the height of the cylinder going to infinity.

Therefore we consider the solution to the problem

$$i\partial_t \Phi_t = H_N \Phi_t$$

$$H_N := \sum_{j=1}^{N} (-\Delta_j + V(r_j)) + \sum_{1 \le j < k \le N} \frac{R_N L_N^3}{N} v\left(L_N(x_j - x_k)\right)$$

acting on the bosonic Hilbert space $\mathcal{H}^{\otimes_s N}$, with Dirichlet boundary conditions; notice that the interacting potential here is **stronger** than in the usual GP scaling.

The different parameters appearing in the Hamiltonian H_N will satisfy

$$\begin{aligned} &\bullet\ h \xrightarrow[N\to+\infty]{} +\infty \text{ (height going to } +\infty) \\ &\bullet\ L_N \xrightarrow[N\to+\infty]{} +\infty,\ L_N \ll N \textbf{ (GP limit)} \\ &\bullet\ a_N := \frac{aR_N}{h} \xrightarrow[N\to+\infty]{} +\infty \textbf{ (TF regime)} \end{aligned} \tag{HP0}$$

Notice that the case discussed at the beginning corresponds to $R_N = 1$ and $L_N = N^\beta$.

For our analysis we will need to consider an initial datum for our many-body system completely factorized and ground state in the z direction of the form

$$\Phi_t(\mathbf{x})\big|_{t=0} = \bigotimes_{j=1}^{N} \frac{1}{\sqrt{h}}\,\varphi(r_j) \tag{HP1}$$

we will also assume that the energy of the initial datum stays close to the ground state energy up to the presence of vortices, i.e.

$$\mathcal{E}^{\mathrm{GP}}[\xi\varphi(\xi y)] \leq E^{\mathrm{GP}} + \mathcal{O}\left(\log\varepsilon\right) \tag{HP2}$$

as said before, this is a crucial hypotheses in [JS15].

Main Result [M. Correggi, DD, P. Pickl]

Under the assumption (HP0) on R_N and (HP1) and (HP2) on the initial datum φ and with v_t such that

$$\begin{cases} i\xi^2 \partial_t v_t = -\Delta v_t + \frac{1}{\varepsilon^2}\left(V(y) + |v_t|^2\right) v_t \\ v_t(y)\big|_{t=0} = \xi \varphi(\xi y) \end{cases} \tag{2.1}$$

with $\xi := a_N^{\frac{1}{s+2}}$ and $\varepsilon := \frac{1}{\sqrt{a_N}}$; let $v_t^{\mathrm{resc}}(x) := \frac{1}{\sqrt{h}\xi} v_t\left(\frac{r}{\xi}\right)$.
Then it exists a time $T_{\max} > 0$ such that for any $0 \le t \le T_{\max}$

$$\lim_{N \to +\infty} \left\| \gamma^{\Phi_t} - |v_t^{\mathrm{resc}}\rangle\langle v_t^{\mathrm{resc}}| \right\| = 0.$$

Remark

An important remark is that the timescale identified by $T_{\max}$ is **larger** than the time needed by vortices to move.

Proceeding as in [P11] one can prove that the evolution of the many-body state is close to the solution of the **Hartree equation**, i.e., if

$$\begin{cases} i\partial\phi_t = -\Delta\phi_t + V(r)\phi_t + R_N\left(L_N^3 v(L_N\cdot) * |\phi_t|^2\right)\phi_t \\ \phi_t(x)|_{t=0} = \frac{1}{\sqrt{h}}\,\varphi(r). \end{cases} \tag{H}$$

then it exists a maximal time $T_{\max}$

$$\lim_{N\to+\infty}\left\|\gamma^{\Phi_t} - |\phi_t\rangle\langle\phi_t|\right\| = 0$$

uniformly in $0 \le t \le T_{\max}$ as $N \to +\infty$ with $T_{\max}$ depending only on ϕ_0, R_N and L_N, where γ^{Φ_t} is the one-body reduced density matrix of Φ_t.

The potential appearing in the Hartree equation goes to a Dirac delta as $N \to +\infty$; this suggests that the solution to the Hartree equation is close to the solution of the **3D GP equation**:

$$\begin{cases} i\partial_t \psi_t = -\Delta \psi_t + V(r)\psi_t + R_N a \left|\psi_t\right|^2 \psi_t \\ \psi_t(x)|_{t=0} = \frac{1}{\sqrt{h}}\, \varphi(r) \end{cases} \tag{3DGP}$$

with $a := \int_{\mathbb{R}^3} dx\ v(x)$.
In particular one can prove L^2 convergence of ϕ_t to ψ_t uniformly in $0 \le t \le T_{\max}$ as $N \to +\infty$.

In order to prove closeness between the Hartree equation and (3DGP) two crucial ingredients are needed:

- the potential $N^{3\beta}v(N^{\beta}x)$ converges as a distribution to a Dirac delta;
- the energy of the solution to (H) stays close to the ground state energy, meaning

$$\mathcal{E}[\phi_t] \leq E + \mathcal{O}\left(R_N^{-\frac{2}{s+2}} \log R_N \right)$$

where

$$\mathcal{E}[\phi] := \int_{\mathbb{R}^2} dy \ \left\{ |\nabla \phi|^2 + V(r)|\phi|^2 + \frac{R_N a}{2}|\phi|^4 \right\}$$

$$E := \inf_{\|\phi\|_2 = 1} \mathcal{E}[\phi].$$

The latter requirement is actually **necessary** to prove [JS15].

Under condition (HP0) the three dimensional solution to the GP problem stays close to the following two dimensional problem

$$\begin{cases} i\partial\varphi_t = -\Delta\varphi_t + V(r)\varphi_t + a_N|\varphi_t|^2\varphi_t \\ \varphi_t(r)|_{t=0} = \varphi(r) \end{cases}$$

where $a_N := \frac{aR_N}{h}$; actually $\psi_t(x) = \frac{1}{\sqrt{h}}\,\varphi_t(r)$ thanks to **the uniqueness of the solution** to the GP equation.

To apply [JS15] we do need now to rescale the dimensions in such a way that the potential and the nonlinearity scale in the same way with N.

Defining $v_t(y) := \xi\varphi_t(r) = \xi\varphi_t(\xi y)$ with $\xi := a_N{}^{\frac{1}{s+2}}$ we get that v_t solves

$$\begin{cases} i\xi^2\partial_t v_t = -\Delta v_t + \frac{1}{\varepsilon^2}\left(V(y) + |v_t|^2\right) v_t \\ v_t(y)\big|_{t=0} = \xi\varphi(\xi y) \end{cases} \tag{2DGPr}$$

where $\varepsilon := \frac{1}{\sqrt{a_N}}$.

Notice that (2DGPr) still needs to be rescaled in time to get to (2DGP), and therefore to the timescale of existence of vortex dynamics.

- In [JS15] it is crucial to be able of relate the **vorticity measure** of a state to the sum of Dirac deltas at the positions of vortices; this requires estimates in strong norms including the energy. One should wonder whether the convergence at the many-body level can be lifted to such energy norms.
- The conditions on L_N needed to prove the result do not allow for the usual Gross-Pitaevskii limit corresponding to $L_N = N^{\beta}$ with $\beta \in [0, 1]$; it is still an open problem to understand whether that but to some logarithmic divergence asymptotic can be reached with the current tecnique.
- One can also consider different traps along the z axis, for example a trap such that $h \to 0$ or a trapping potential with spectral gap going to infinity; in the first case our analysis applies.

Thanks for the attention!

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