Dynamics of Bose-Einstein Condensates in the Thomas-Fermi Regime

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The problem

We consider the many-body dynamics for a system of trapped bosons in the Thomas-Fermi regime. The Hamiltonian is given by

$$H_N := \sum_{j=1}^{N} \left[-\Delta_j + V(\mathbf{x}_j)\right] + \sum_{1\le j<k\le N} g_N N^{3\beta-1} v\left(N^{\beta}(\mathbf{x}_j - \mathbf{x}_k)\right),$$

and if the initial datum for the system is the state Ψ_0, the evolved system at time t is described by the state $\Psi_t := e^{-itH_N}\Psi_0$.

If the initial datum exhibits Bose-Einstein condensation (BEC) on the state ψ_0, meaning that

$$\gamma^{(1)}_{\Psi_0} \to |\psi_0\rangle\langle\psi_0|,$$

where $\gamma^{(1)}_{\Psi_0}$ is the one-particle reduced density matrix, then it is expected that at time t the state still exhibit BEC on a state ψ_t, with ψ_t solving the Gross-Pitaevskii equation

$$\begin{cases} i\partial_t\psi_t = -\Delta\psi_t + V\Psi_t + g_N\left|\psi_t\right|^2\psi_t \\ \psi_t|_{t=0} = \psi_0. \end{cases}$$

Our result is that under suitable assumptions on the system this is indeed true.

Setting

- The one-particle Hilbert space is $\mathfrak{h} := L^2\left(\mathbb{R}^3\right)$ and the many-body bosonic Hilbert space is $H_N := \mathfrak{h}^{\otimes_s N}$;
- $v \in L^\infty$ is spherically symmetric and compactly supported with $\int_{\mathbb{R}^3} d\mathbf{x}\, v(\mathbf{x})$;
- V is an anharmonic radial homogeneous potential, i.e. $V(r) = k|r|^s$, with $k>0$ and $s>2$;
- g_N go to infinity as N goes to infinity and $g_N \ll \log N$.

Rescaling

To prove our result it is useful to rescale dimensions. We apply a unitary trasformation U on the one-particle state $\psi\in\mathfrak{h}$ defined as

$$(U\psi)(\mathbf{y}) := g_N^{-\frac{3}{s+3}}\psi\left(g_N^{-\frac{1}{s+3}}\mathbf{y}\right) =: \phi(\mathbf{y}).$$

This induces a similar unitary transformation on $\mathcal{H}_N$ that we call U_N. The unitary transform of the Hamiltonian now shows that in the rescale dynamics the kinetic term is depleted by a small parameter ε appearing in front of the laplacian. More explicitly, if $K_N := g_N^{-s/(s+3)} U_N H U_N^*$, then

$$K_N = \sum_{j=1}^{N}\left[-\varepsilon^2\Delta_j + V(\mathbf{y}_j)\right] + \frac{1}{N}\sum_{1\le j<k\le N} \widetilde{N}^{3\beta} v\left(\widetilde{N}^{\beta}(\mathbf{y}_j - \mathbf{y}_k)\right),$$

with $\varepsilon := g_N^{-(s+2)/2(s+3)}$ and $\widetilde{N} := \varepsilon^{-2/\beta(s+2)}N$.

From Many-Body to Hartree

Applying now a similar analysis as in [P] one can prove that the evolution of the many-body state is close to the solution of the Hartree equation, i.e., if

$$\begin{cases} i\partial\varphi_t = -\varepsilon^2\Delta\varphi_t + V\varphi_t + \left(\widetilde{N}^{3\beta} v\left(\widetilde{N}^{\beta}\cdot\right) * |\varphi_t|^2\right)\varphi_t \\ \varphi_t|_{t=0} = \phi_0 := U\psi_0. \end{cases} \tag{H}$$

then

$$\lim_{N\to+\infty}\left\|\gamma^{\Phi_t} - |\varphi_t\rangle\langle\varphi_t|\right\|_{\mathrm{op}} = 0$$

for any given time $t>0$. To proceed as in [P], however, we need the following conjecture.

Conjecture. *If* $\|\phi_0\|_\infty = \mathcal{O}(1)$, *then* $\sup_t\|\phi_t\|_\infty = \mathcal{O}(1)$.

The content of the Conjecture does not trivially follows from the properties of Equation (H). Although it being a crucial requirement for our result, this is mostly a technical problem.

From Hartree to Gross-Pitaevskii

The next step is to prove closeness of φ_t to the solution of the GP equation:

$$\begin{cases} i\partial_t\phi_t = -\varepsilon^2\Delta\phi_t + V\phi_t + |\phi_t|^2\phi_t \\ \phi_t|_{t=0} = \phi_0 \end{cases} \tag{3DGP}$$

In particular we can prove L^2 convergence of φ_t to ϕ_t for any time t, but to do so we need two crucial ingredients:

- the potential $N^{3\beta}v\left(N^{\beta}\mathbf{x}\right)$ converges as a distribution to a Dirac delta;
- starting with a state with low energy, also φ_t and ϕ_t have low energy; more precisely, let

$$\mathcal{E}[\phi] := \int_{\mathbb{R}^2} dy\,\left\{|\nabla\phi|^2 + V(r)|\phi|^2 + \frac{g_N}{2}|\varphi|^4\right\}$$
$$E := \inf_{\|\varphi\|_2=1}\mathcal{E}[\phi].$$

If we assume that the energy of the initial datum satisfies

$$\mathcal{E}[\phi_0] \le E + \mathcal{O}(\varepsilon|\log\varepsilon|). \tag{E}$$

then also ϕ_t and φ_t satify the same inequality.

Remark. *Notice that the condition (E) is not too restrictive; in particular the derivation of the Thomas-Fermi regime is of particular relevance in the study of vortex motion in a superfluid. In this case the system is described by a two-dimensional problem analogous to equation (3DGP), as one can see for example in [JS]. In that very work, a precise study of the motion of vortices is carried on, and one can easily see that the hypotheses there contained imply in particular condition (E).*

Main Result

Theorem. *Assume the Conjecture to be true; moreover, let there be BEC in the initial datum on the state* ψ_0, *i.e. suppose*

$$\lim_{N\to+\infty}\left\|\gamma^{\Psi_0} - |\varphi_t\rangle\langle\psi_0|\right\|_{\mathrm{op}} = 0$$

with $\phi_0 := U\psi_0$ *satisfying condition (E).*

Then for any $\beta\in[0,1/6]$ *there is still BEC at time* t *on the state* $\psi_t := U^*\phi_t$, *with* ϕ_t *satisfying equation (3DGP), i.e.*

$$\lim_{N\to+\infty}\left\|\gamma^{\Psi_t} - |\psi_t\rangle\langle\psi_t|\right\|_{op} = 0.$$

Open Questions

In [JS] it is crucial to be able of relate the **measure of the vorticity** of a state to a Dirac delta representing the position of the vortices; this involves a different topology on the GP state. One should wonder how to translate this topology at the level of the many-body system; our current idea is to consider a norm that contains information about the energy of the state too.

References

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